ANALYTIC TORSION FOR CALABI-YAU THREEFOLDS

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Abstract. After Bershadsky-Cecotti-Ooguri-Vafa, we introduce an invariant of Calabi-Yau threefolds, which we call the BCOV invariant and which we obtain using analytic torsion. We give an explicit formula for the BCOV invariant as a function on the compactified moduli space, when it is isomorphic to a projective line. As a corollary, we prove the formula for the BCOV invariant of quintic mirror threefolds conjectured by Bershadsky-Cecotti-Ooguri-Vafa.

Contents

1. Introduction

In the outstanding papers [6], [7], Bershadsky-Cecotti-Ooguri-Vafa made a deep study of the generating function F_g of genus-g Gromov-Witten invariants for Calabi-Yau threefolds. One mathematical surprise, which they obtained from physical arguments, is a system of holomorphic anomaly equations satisfied by the functions F_g, $g \geq 1$. From the holomorphic anomaly equations, they obtained a conjectural explicit formula for F_g of a general quintic threefolds in $\mathbb{P}^4$ and thus they extended the mirror symmetry conjecture of Candelas-de la Ossa-Green-Parkes [14].

By focusing on the genus-1 holomorphic anomaly equation, they conjectured that F_1 of a Calabi-Yau threefold is expressed as a certain linear combination of the Ray-Singer analytic torsions (cf. [11], [46]) of its mirror Calabi-Yau threefolds. After Bershadsky-Cecotti-Ooguri-Vafa, we call the linear combination of Ray-Singer analytic torsions in [7] the BCOV torsion, which is the main subject of this paper.

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By making use of the curvature formula for Quillen metrics [11], Bershadsky-Cecotti-Ooguri-Vafa obtained a variational formula for the BCOV torsion of Ricci-flat Calabi-Yau manifolds [7]. Fang-Lu [17] expressed the variation of the BCOV torsion of Ricci-flat Calabi-Yau manifolds as a linear combination of the Weil-Petersson metric [53] and the generalized Hodge metrics [36], which led them to some new results on the moduli space of polarized Calabi-Yau manifolds.

On the other hand, as a consequence of the duality in string theory, Harvey-Moore [25] conjectured that the BCOV torsion of certain Ricci-flat Calabi-Yau threefolds is expressed as the product of the norms of the Borcherds Φ-function [13] and the Dedekind η-function. Their conjecture was proved by Yoshikawa [60]. In his proof, an invariant of $K3$ surfaces with involution, which he obtained using equivariant analytic torsion [8] and a Bott-Chern class [11], played a crucial role.

In this paper, we extend the constructions of Bershadsky-Cecotti-Ooguri-Vafa and Yoshikawa to introduce a new invariant of Calabi-Yau threefolds, which we call the BCOV invariant, and we get an explicit formula for the BCOV invariant as a function on the compactified moduli space when it is isomorphic to $\mathbb{P}^1$. As a corollary of our formula, we prove one part of the conjecture of Bershadsky-Cecotti-Ooguri-Vafa concerning the BCOV torsion of quintic mirror threefolds. Let us explain our results in more details.

Let X be a Calabi-Yau threefold. Let g be a Kähler metric on X with Kähler form γ. We set $\overline{X} = (X, \gamma)$. Let $\tau(\overline{X}, \overline{\Omega}^p_X)$ be the Ray-Singer analytic torsion of $\Omega^p_X = \wedge^p T^*X$ with respect to g. We define the BCOV torsion of $\overline{X}$ as

$$\mathcal{T}_{\mathrm{BCOV}}(\overline{X}) = \prod_{p\geq 0} \tau(\overline{X}, \overline{\Omega}^p_X)^{(-1)^p p}.$$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{b_2(X)}\}$ be an integral basis of $H^2(X,\mathbb{Z})/\mathrm{Torsion}$. By Hodge theory and the Lefschetz decomposition theorem, $H^2(X,\mathbb{R})$ is equipped with the L^2-metric $\langle\cdot,\cdot\rangle_{L^2,[\gamma]}$, which depends only on the Kähler class $[\gamma]$. We define

$$\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma]) = \det\left(\langle \mathbf{e}_i, \mathbf{e}_j\rangle_{L^2,[\gamma]}\right)_{1\leq i,j\leq b_2(X)},$$

which is independent of the choice of an integral basis of $H^2(X,\mathbb{Z})/\mathrm{Torsion}$.

Let η be a nowhere vanishing holomorphic 3-form on X. Let $c_3(X,\gamma)$ be the top Chern form of (TX, g). We set $\mathrm{Vol}(X,\gamma) = (2\pi)^{-3}\int_X \gamma^3$ and $\chi(X) = \int_X c_3(X,\gamma)$. We define

$$\mathcal{A}(\overline{X}) = \mathrm{Vol}(X,\gamma)^{\frac{\chi(X)}{12}} \exp\left[-\frac{1}{12}\int_X \log\left(\frac{\sqrt{-1}\eta\wedge\bar{\eta}}{\gamma^3/3!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|^2_{L^2}}\right) c_3(X,\gamma)\right],$$

which is independent of the choice of η. We define the real number $\tau_{\mathrm{BCOV}}(X)$ as

$$\tau_{\mathrm{BCOV}}(X) = \mathrm{Vol}(X,\gamma)^{-3}\,\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma])^{-1}\,\mathcal{A}(\overline{X})\,\mathcal{T}_{\mathrm{BCOV}}(\overline{X}).$$

In Sect. 4.4, we show that $\tau_{\mathrm{BCOV}}(X)$ is independent of the choice of γ. Hence $\tau_{\mathrm{BCOV}}(X)$ is an invariant of X, which we call the BCOV invariant. The purpose of this paper is to study τ_{BCOV} as a function on the moduli of Calabi-Yau threefolds.

Let $\mathcal{X}$ be a (possibly singular) irreducible projective fourfold. Let $\pi\colon \mathcal{X}\to\mathbb{P}^1$ be a surjective flat morphism with discriminant locus $\mathcal{D}$. Let ψ be the inhomogeneous coordinate of $\mathbb{P}^1$, and set $X_\psi := \pi^{-1}(\psi)$ for $\psi\in\mathbb{P}^1$. We assume the following:
(i) $\mathcal{D}$ is a finite subset of $\mathbb{P}^1$ such that $\infty\in\mathcal{D}$ and $\mathcal{D}\setminus\{\infty\}\neq\emptyset$;
(ii) X_ψ is a Calabi-Yau threefold with $h^2(\Omega^1_{X_\psi}) = 1$ for $\psi\in\mathbb{P}^1\setminus\mathcal{D}$;

(iii) $\mathrm{Sing}\, X_\psi$ consists of a unique ordinary double point (ODP) for $\psi \in \mathcal{D} \setminus \{\infty\}$;
(iv) $\mathrm{Sing}(\mathcal{X}) \cap X_\infty = \emptyset$ and X_∞ is a divisor of normal crossing.

Under these assumptions, the relative dualizing sheaf $K_{\mathcal{X}/\mathbb{P}^1}$ is locally free on $\mathcal{X}$, and its direct image sheaf $\pi_* K_{\mathcal{X}/\mathbb{P}^1}$ is locally free on $\mathbb{P}^1$.

For $\psi \in \mathbb{P}^1 \setminus \{\infty\}$, let $(\mathrm{Def}(X_\psi), [X_\psi])$ be the Kuranishi space of X_ψ. Since X_ψ is Calabi-Yau, $\dim \mathrm{Def}(X_\psi) = 1$. We identify $(\mathrm{Def}(X_\psi), [X_\psi])$ with $(\mathbb{C}, 0)$ by the smoothness of the Kuranishi space (cf. [53], [54], [55]). Let $\mu_\psi \colon (\mathbb{P}^1, \psi) \to (\mathrm{Def}(X_\psi), [X_\psi])$ be the map of germs that induces the family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ near ψ. The ramification index $r(\psi)$ of $\pi\colon \mathcal{X} \to \mathbb{P}^1$ at $\psi \in \mathbb{P}^1$ is defined as the vanishing order of μ_ψ at ψ. Let $\{R_j\}_{j \in J}$ be the set of points of $\mathbb{P}^1$ with ramification index > 1, and write $\mathcal{D} \setminus \{\infty\} = \{D_k\}_{k \in K}$. We set $r_j = r(R_j)$ for $j \in J$ and $r_k = r(D_k)$ for $k \in K$.

Outside $\mathcal{D} \cup \{R_j\}_{j \in J}$, $T\mathbb{P}^1$ is equipped with the Weil-Petersson metric. Let $\|\cdot\|$ be the singular Hermitian metric on $(\pi_* K_{\mathcal{X}/\mathbb{P}^1})^{\otimes(48+\chi)} \otimes (T\mathbb{P}^1)^{\otimes 12}$ induced from the L^2-metric on $\pi_* K_{\mathcal{X}/\mathbb{P}^1}$ and from the Weil-Petersson metric on $T\mathbb{P}^1$.

Main Theorem 1.1. *Let Ξ be a meromorphic section of $\pi_* K_{\mathcal{X}/\mathbb{P}^1}$ with*

$$\mathrm{div}(\Xi) = \sum_{i \in I} m_i\, P_i + m_\infty\, P_\infty, \qquad P_i \neq P_\infty\ (i \in I).$$

Identify the points P_i, R_j, D_k with their coordinates $\psi(P_i), \psi(R_j), \psi(D_k) \in \mathbb{C}$, respectively. Set $\chi = \chi(X_\psi)$, $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$. Then there exists $C \in \mathbb{R}_{>0}$ such that

$$\tau_{\mathrm{BCOV}}(X_\psi) = C \left\| \prod_{i \in I, j \in J, k \in K} \frac{(\psi - D_k)^{2r_k}}{(\psi - P_i)^{(48+\chi)m_i} (\psi - R_j)^{12(r_j - 1)}}\, \Xi_\psi^{48+\chi} \otimes \left(\frac{\partial}{\partial \psi}\right)^{12} \right\|^{\frac{1}{6}}.$$

As a corollary of the Main Theorem 1.1, we give a partial answer to the conjecture of Bershadsky-Cecotti-Ooguri-Vafa, which we explain briefly (cf. Sect. 12).

Let $p\colon \mathcal{X} \to \mathbb{P}^1$ be the pencil of quintic threefolds in $\mathbb{P}^4$ defined by

$$\mathcal{X} := \{([z], \psi) \in \mathbb{P}^4 \times \mathbb{P}^1;\ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4 = 0\}, \quad p = \mathrm{pr}_2.$$

Let $\mathbb{Z}_5$ be the set of fifth roots of unity and define

$$G := \{(a_0, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_5)^5;\ a_0 a_1 a_2 a_3 a_4 = 1\} / \mathbb{Z}_5(1,1,1,1,1) \cong \mathbb{Z}_5^3.$$

We regard G as a group of projective transformations of $\mathbb{P}^4$. Since G preserves the fibers of p, we have the induced family $p\colon \mathcal{X}/G \to \mathbb{P}^1$. Let $\mathcal{D}$ be the discriminant locus of the family $p\colon \mathcal{X} \to \mathbb{P}^1$. By [4], [39], there exists a resolution $q\colon \mathcal{W} \to \mathcal{X}/G$ such that $W_\psi = q^1(X_\psi)$ is a smooth Calabi-Yau threefold for $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$ and such that $\mathrm{Sing}\, W_\psi$ consists of a unique ODP if $\psi^5 = 1$. The family of Calabi-Yau threefolds $\pi\colon \mathcal{W} \to \mathbb{P}^1$ is called a family of quintic mirror threefolds.

After Candelas-de la Ossa-Green-Parkes [14], $\pi_* K_{\mathcal{W}/\mathbb{P}^1}$ and $T\mathbb{P}^1$ are trivialized as follows near $\psi = \infty$. For $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$, we define a holomorphic 3-form on X_ψ by

$$\Omega_\psi = \left(\frac{2\pi\sqrt{-1}}{5}\right)^{-3} 5\psi\, \frac{z_4\, dz_0 \wedge dz_1 \wedge dz_2}{\partial F_\psi(z)/\partial z_3}.$$

Since Ω_ψ is G-invariant, Ω_ψ induces a holomorphic 3-form on X_ψ/G in the sense of orbifolds. We identify Ω_ψ with the corresponding holomorphic 3-form on X_ψ/G,

and we define a holomorphic 3-form Ξ_ψ on W_ψ as $\Xi_\psi = q_\psi^* \Omega_\psi$. We define

$$y_0(\psi) = \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \qquad |\psi| > 1.$$

Then $\pi_* K_{\mathcal{W}/\mathbb{P}^1}$ is trivialized by the local section $\Xi_\psi / y_0(\psi)$ near $\psi = \infty$.

Let q be the coordinate of the unit disc in $\mathbb{C}$. We identify the parameters ψ^5 and q via the mirror map [14]. Then $T\mathbb{P}^1$ is trivialized by the local section $q\, d/dq = q\,(d\psi/dq)\, d/d\psi$ near $\psi = \infty$. (See Sect. 12.)

We define a multi-valued analytic function $F_{1,B}^{\rm top}(\psi)$ near $\infty \in \mathbb{P}^1$ as

$$F_{1,B}^{\rm top}(\psi) = \left(\frac{\psi}{y_0(\psi)}\right)^{\frac{62}{3}} (\psi^5 - 1)^{-\frac{1}{6}}\, q\, \frac{d\psi}{dq}$$

and a power series in q as $F_{1,A}^{\rm top}(q) = F_{1,B}^{\rm top}(\psi(q))$. The conjectures of Bershadsky-Cecotti-Ooguri-Vafa [6], [7] can be formulated as follows:

Conjecture 1.2. **(A)** *Let $N_g(d)$ be the genus-g Gromov-Witten invariant of degree d of a general quintic threefold in $\mathbb{P}^4$. Then*

$$q\frac{d}{dq} \log F_{1,A}^{\rm top}(q) = \frac{50}{12} - \sum_{n,d=1}^{\infty} N_1(d)\, \frac{2nd\, q^{nd}}{1 - q^{nd}} - \sum_{d=1}^{\infty} N_0(d)\, \frac{2d\, q^d}{12(1 - q^d)}.$$

(B) *The following identity holds near $\psi = \infty$:*

$$\tau_{\rm BCOV}(W_\psi) = \text{Const.} \left\| \frac{1}{F_{1,B}^{\rm top}(\psi)^3} \left(\frac{\Xi_\psi}{y_0(\psi)}\right)^{62} \otimes \left(q \frac{d}{dq}\right)^3 \right\|^{\frac{2}{3}}.$$

In Sect. 12, we prove the following:

Theorem 1.3. *The Conjecture 1.2 (B) holds.*

For the remaining Conjecture 1.2 (A), see Li-Zinger [34]. In [18], we shall study the BCOV invariant of Calabi-Yau threefolds with higher dimensional moduli and the BCOV torsion of Calabi-Yau manifolds of dimension greater than 3.

Let us briefly explain our approach to prove the Main Theorem 1.1. We follow the approach in [60]. Let $\Omega_{\rm WP}$ be the Weil-Petersson form on $\mathbb{P}^1 \setminus \mathcal{D}$, and let $\mathrm{Ric}\,\Omega_{\rm WP}$ be the Ricci-form of $\Omega_{\rm WP}$. By [36], [37], the $(1,1)$-forms $\Omega_{\rm WP}$ and $\mathrm{Ric}\,\Omega_{\rm WP}$ have Poincaré growth on $\mathbb{P}^1 \setminus \mathcal{D}$, so that they extend trivially to closed positive $(1,1)$-currents on $\mathbb{P}^1$ (cf. Sect. 7.3). We identify $\Omega_{\rm WP}$ and $\mathrm{Ric}\,\Omega_{\rm WP}$ with their trivial extensions. For a divisor D on $\mathbb{P}^1$, let δ_D denote the Dirac δ-current on $\mathbb{P}^1$ associated to D. Regard $\tau_{\rm BCOV}$ as a function on $\mathbb{P}^1 \setminus \mathcal{D}$. By making use of the Poincaré-Lelong formula, the Main Theorem 1.1 is deduced from the following:

Claim 1.4. *Set $\mathcal{D}^* = \sum_{k\in K} r_k\, D_k$. Then there exists $a \in \mathbb{R}$ such that*

$$dd^c \log \tau_{\rm BCOV} = -\left(\frac{\chi}{12} + 4\right) \Omega_{\rm WP} - \mathrm{Ric}\,\Omega_{\rm WP} + \frac{1}{6}\, \delta_{\mathcal{D}^*} + a\, \delta_\infty. \tag{1.1}$$

We shall establish Claim 1.4 as follows:

(a) By making use of the curvature formula for Quillen metrics of Bismut-Gillet-Soulé [11], we prove the variational formula like (1.1) for an arbitrary family of Calabi-Yau threefolds. As a result, we get Eq. (1.1) on the open part $\mathbb{P}^1 \setminus \mathcal{D}$. More precisely, we introduce a Hermitian line, called the BCOV Hermitian line, for

an arbitrary Calabi-Yau manifold of arbitrary dimension, which we obtain using determinants of cohomologies [28], Quillen metrics [11], [44], and a Bott-Chern class like $\mathcal{A}(\cdot)$. Then the BCOV Hermitian line of a Calabi-Yau manifold depends only on the complex structure of the manifold. The Hodge diamond of Calabi-Yau threefolds are so simple that the BCOV Hermitian line reduces to the scalar invariant $\tau_{\rm BCOV}$ in the case of threefolds. Hence Eq. (1.1) on $\mathbb{P}^1 \setminus \mathcal{D}$ is deduced from the curvature formula for the BCOV Hermitian line bundles. (See Sect. 4).

(b) To establish the formula for $\log \tau_{\rm BCOV}$ near $\mathcal{D}$, we fix a specific holomorphic extension of the BCOV bundle from $\mathbb{P}^1 \setminus \mathcal{D}$ to $\mathbb{P}^1$, which we call the Kähler extension. (See Sect. 5.) Since $\tau_{\rm BCOV}$ is the ratio of the Quillen metric and the L^2-metric on the BCOV bundle, it suffices to determine the singularities of the Quillen metric and the L^2-metric on the extended BCOV bundle. We determine the singularity of the Quillen metric on the extended BCOV bundle with respect to the metric on $T\mathcal{X}/\mathbb{P}^1$ induced from a Kähler metric on $\mathcal{X}$. The anomaly formula for Quillen metrics of Bismut-Gillet-Soulé [11] and a formula for the singularity of Quillen metrics [9], [61] play the central role. (See Sect. 5.).

(c) By the smoothness of $\mathrm{Def}(X_\psi)$ at $\psi \in \mathcal{D}^*$ [26], [45], [54], the behavior of the L^2-metric on the extended BCOV bundle near $\mathcal{D}^*$ is determined by the singularity of $\Omega_{\rm WP}$ near $\mathcal{D}^*$, which was computed by Tian [54]. (See Sects. 6, 7, 8.) To determine the behavior of the L^2 metric on the extended BCOV bundle at $\psi = \infty$, one may assume that $\pi\colon \mathcal{X} \to \mathbb{P}^1$ is semi-stable at $\psi = \infty$ by Mumford [27]. We consider another holomorphic extension of the BCOV bundle, i.e., the canonical extension in Hodge theory [48]. With respect to the canonical extension, the L^2-metric has at most an algebraic singularity at $\psi = \infty$ by Schmid [48]. Comparing the two extensions, we show that the L^2-metric has at most an algebraic singularity at $\psi = \infty$ with respect to the Kähler extension. (See Sect. 9.) By the residue theorem and assumption (ii), the number a in Eq. (1.1) is determined by the degrees of the divisors $\mathcal{D}^*$, $\mathrm{div}(\Xi)$, $\sum_{j \in J}(r_j - 1)\, R_j$. (See Sect. 11.)

This paper is organized as follows. In Sect. 2, we recall the deformation theory of Calabi-Yau threefolds. In Sect. 3, we recall the definition of Quillen metrics and the corresponding curvature formula. In Sect. 4, we introduce the BCOV invariant and prove its variational formula. In Sect. 5, we study the boundary behavior of Quillen metrics. In Sect. 6, we study the boundary behavior of Kodaira-Spencer map. In Sect. 7, we study the boundary behavior of the Weil-Petersson metric and the Hodge metric. In Sects. 8 and 9, we study the boundary behavior of the BCOV invariant. In Sect. 10, we extend the variational formula for the BCOV invariant to the boundary of moduli space. In Sect. 11, we prove the Main Theorem. In Sect. 12, we study a conjecture of Bershadsky-Cecotti-Ooguri-Vafa. In Sect. 13, we study a conjecture of Harvey-Moore.

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2. Calabi-Yau varieties with at most one ordinary double point

2.1. Calabi-Yau varieties with at most one ODP and their deformations

2.1.1. *Calabi-Yau varieties with at most one ODP.* Recall that an n-dimensional singularity is an *ordinary double point* (*ODP* for short) if it is isomorphic to the hypersurface singularity at $0 \in \mathbb{C}^n$ defined by the equation $z_0^2 + \cdots + z_n^2 = 0$.

Definition 2.1. A complex projective variety X of dimension $n \geq 3$ satisfying the following conditions is called a *Calabi-Yau n-fold with at most one ODP*:
(i) There exists a nowhere vanishing holomorphic n-form on $X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$;
(ii) X is connected and $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < n$;
(iii) The singular locus $\mathrm{Sing}(X)$ consists of empty or at most one ODP.

Throughout this paper, we use the following notation: For a complex space Y, let Θ_Y be the tangent sheaf of Y, let Ω_Y^1 be the sheaf of Kähler differentials on Y, and let K_Y be the dualizing sheaf of Y. The sheaf Ω_Y^p is defined as $\bigwedge^p \Omega_Y^1$. On the regular part of Y, the sheaves Θ_Y, Ω_Y^p, K_Y are often identified with the corresponding holomorphic vector bundles TY, $\bigwedge^p T^*Y$, $\det T^*Y$, respectively.

We set $\varDelta(r) := \{t \in \mathbb{C};\, |t| < r\}$ and $\varDelta(r)^* := \varDelta(r) \setminus \{0\}$ for $r > 0$. We write $\varDelta$ (resp. $\varDelta^*$) for $\varDelta(1)$ (resp. $\varDelta(1)^*$).

Since an ODP is a hypersurface singularity, the dualizing sheaf of a Calabi-Yau n-fold with at most one ODP is trivial by (i).

2.1.2. *Deformations of Calabi-Yau varieties with at most one ODP.* Let X be a Calabi-Yau n-fold with at most one ODP.

Definition 2.2. Let $(S, 0)$ be a complex space with marked point and let $\mathcal{X}$ be a complex space. A proper, surjective, flat holomorphic map $\pi\colon \mathcal{X} \to S$ is called a *deformation* of X if $\pi^{-1}(0) \cong X$. If $\mathcal{X}$ and S are smooth and if a general fiber of $\pi\colon \mathcal{X} \to S$ is smooth, the deformation $\pi\colon (\mathcal{X}, X) \to (S, 0)$ is called a *smoothing* of X. If there exists a smoothing of X, X is said to be *smoothable*.

We refer to [40, Example 5.8] for an example of a non-smoothable Calabi-Yau threefold with a unique ODP as its singular set.

Since $H^0(X, \Theta_X) = 0$ (cf. [40, pp.432, l.23]), there exists a deformation germ $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def(X)}, [\mathrm{X}])$ of X with the universal property: Every deformation germ $\pi\colon (\mathcal{X}, X) \to (S, 0)$ is induced from $\mathfrak{p}\colon \mathfrak{X} \to \mathrm{Def(X)}$ by a unique holomorphic map $f\colon (S, 0) \to (\mathrm{Def(X)}, [\mathrm{X}])$. This local universal deformation of X is called the *Kuranishi family* of X. The Kuranishi family is unique up to an isomorphism. The base space $(\mathrm{Def(X)}, [\mathrm{X}])$ is called the *Kuranishi space* of X. By [26], [45], [53], [54], [55], $\mathrm{Def}(X)$ is smooth. We denote by $T_{\mathrm{Def}(X),[X]}$ the tangent space of $\mathrm{Def}(X)$ at $[X]$. See [16], [22], [32] for more details about the Kuranishi family.

For a deformation $\pi\colon (\mathcal{X}, X) \to (S, 0)$, the fiber X_s ($s \in S$) is a Calabi-Yau n-fold with at most one ODP if $s \in S$ is sufficiently close to 0 (cf. [40, Prop. 6.1], [54, Prop. 4.2]).

In the rest of Subsection 2.1, we assume that X is a smoothable Calabi-Yau n-fold with at most one ODP. Let $\pi\colon (\mathcal{X}, X) \to (S, 0)$ be a smoothing. The critical locus of π is defined by
$$\Sigma_\pi := \{x \in \mathcal{X};\, d\pi_x = 0\}.$$
The discriminant locus of $\pi\colon \mathcal{X} \to S$ is the subvariety of S defined by
$$\mathcal{D} := \pi(\Sigma_\pi) = \{s \in S;\, \mathrm{Sing}(X_s) \neq \emptyset\}.$$

Lemma 2.3. *Let $N+1=\dim S$. For $p\in\mathrm{Sing}(X)$, there exists a neighborhood $V_p\cong\Delta^{n+1}\times\Delta^N$ of p in $\mathcal{X}$ such that*

$$\pi|_{V_p}(z,w)=(z_0^2+\cdots+z_n^2,w_1,\ldots,w_N),\qquad z=(z_0,\ldots,z_n),\quad w=(w_1,\ldots,w_N).$$

In particular, if $\mathrm{Sing}(X)\neq\emptyset$, $\mathcal{D}$ is a divisor of S smooth at 0.

Proof. Let $p\in\mathrm{Sing}(X)$. Let $s=(s_0,\ldots,s_N)$ be a system of coordinates near $0\in S$. By e.g. [33, pp.103, (6.7)], there exists $f_p\in\mathcal{O}_{S,0}$ such that

$$\mathcal{O}_{\mathcal{X},p}\cong\mathcal{O}_{\mathbb{C}^{n+1}\times S,(0,0)}/(z_0^2+\cdots+z_n^2+f_p(s)),\qquad\pi(z,s)=s.$$

Since $\mathcal{X}$ is smooth, we get $df_p(0)\neq0$. Hence we can assume that $f_p(s)=s_0$ after a suitable change of the coordinates of S. □

2.1.3. *The Kodaira-Spencer map.* For a smoothing $\pi\colon(\mathcal{X},X)\to(S,0)$, the short exact sequence of sheaves on X

$$0\longrightarrow\pi^*\Omega^1_S|_X\longrightarrow\Omega^1_{\mathcal{X}}|_X\longrightarrow\Omega^1_X\longrightarrow0$$

induces the long exact sequence:

$$\cdots\longrightarrow\mathrm{Hom}_{\mathcal{O}_{\mathrm{X}}}(\pi^*\Omega^1_{\mathrm{S}}|_{\mathrm{X}},\mathcal{O}_{\mathrm{X}})\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{X}}}(\Omega^1_{\mathrm{X}},\mathcal{O}_{\mathrm{X}})\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{X}}}(\Omega^1_{\mathcal{X}}|_{\mathrm{X}},\mathcal{O}_{\mathrm{X}})\longrightarrow\cdots$$

Definition 2.4. The *Kodaira-Spencer map* of $\pi\colon(\mathcal{X},X)\to(S,0)$ is the coboundary map

$$\rho_0\colon T_0S=\mathrm{Hom}_{\mathcal{O}_{\mathrm{X}}}(\pi^*\Omega^1_{\mathrm{S}}|_{\mathrm{X}},\mathcal{O}_{\mathrm{X}})\to\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{X}}}(\Omega^1_{\mathrm{X}},\mathcal{O}_{\mathrm{X}}).$$

Proposition 2.5. *The Kodaira-Spencer map $\rho_{[X]}\colon T_{\mathrm{Def}(X),[X]}\to\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ for the Kuranishi family of X is an isomorphism.*

Proof. See [26], [45], [53], [54], [55]. □

Let

$$r\colon\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)\ni\alpha\to\alpha|_{X_{\mathrm{reg}}}\in\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_{\mathrm{reg}}},\mathcal{O}_{X_{\mathrm{reg}}})=H^1(X_{\mathrm{reg}},\Theta_X)$$

be the restriction map. Since $n\geq3$, $r\colon\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)\to H^1(X_{\mathrm{reg}},\Theta_X)$ is an isomorphism by [47, Th. 2] and [54, Prop. 1.1].

Lemma 2.6. *Under the natural identification $H^0(X_{\mathrm{reg}},\pi^*\Theta_S|_{X_{\mathrm{reg}}})\cong T_0S$ via π, the composition $r\circ\rho_0\colon T_0S\to H^1(X_{\mathrm{reg}},\Theta_X)$ is the coboundary map of the long exact sequence of cohomologies associated with the short exact sequence of sheaves*

$$0\longrightarrow\Theta_{X_{\mathrm{reg}}}\longrightarrow\Theta_{\mathcal{X}}|_{X_{\mathrm{reg}}}\longrightarrow\pi^*\Theta_S|_{X_{\mathrm{reg}}}\longrightarrow0.\tag{2.1}$$

Proof. The commutative diagram of the short exact sequences of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega^1_S|_X \arrow[r]\arrow[d,"r"'] & \Omega^1_{\mathcal{X}}|_X \arrow[r]\arrow[d,"r"'] & \Omega^1_X \arrow[r]\arrow[d,"r"'] & 0\\
0 \arrow[r] & \pi^*\Omega^1_S|_{X_{\mathrm{reg}}} \arrow[r] & \Omega^1_{\mathcal{X}}|_{X_{\mathrm{reg}}} \arrow[r] & \Omega^1_X|_{X_{\mathrm{reg}}} \arrow[r] & 0
\end{tikzcd}$$

induces the commutative diagram of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\pi^*\Omega^1_S|_X,\mathcal{O}_X) \arrow[r]\arrow[d,"r"'] & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X) \arrow[r]\arrow[d,"r"'] & {}\\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\pi^*\Omega^1_S|_{X_{\mathrm{reg}}},\mathcal{O}_{X_{\mathrm{reg}}}) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_{\mathrm{reg}}},\mathcal{O}_{X_{\mathrm{reg}}}) \arrow[r] & {}
\end{tikzcd}$$

where the first (resp. second) vertical arrow is isomorphic by the normality of $\mathrm{Sing}(X)$ (resp. [54, Prop. 1.2]). Since $\pi^*\Omega^1_S|_{X_{\mathrm{reg}}}$, $\Omega^1_{\mathcal{X}}|_{X_{\mathrm{reg}}}$, $\Omega^1_{X_{\mathrm{reg}}}$ are locally free, the second line is the long exact sequence of cohomologies associated with (2.1). $\square$

Lemma 2.7. *Suppose X is smoothable. Then the Kuranishi family of X is a smoothing of X.*

Proof. Since the assertion is obvious when X is smooth, we assume that X has a unique ODP p. Since X is smoothable, a general fiber of the Kuranishi family of X is smooth. We must prove the smoothness of the total space $\mathfrak{X}$ of the Kuranishi family of X. Since $\mathrm{Sing}(X) = \{p\}$, it suffices to prove the smoothness of $\mathfrak{X}$ at p.

Let $\mathrm{Def}(X,p) \cong (\mathbb{C},0)$ be the Kuranishi space of the ODP (X,p) (cf. [33, Chap. 6 C]). The universal deformation of X induces a holomorphic map of germs $f\colon \mathrm{Def}(X) \to \mathrm{Def}(X,p)$. The existence of a smoothing of X implies the surjectivity of the differential of f at $[X]$. Hence f may be regarded as a part of a system of coordinates of $\mathrm{Def}(X)$ at $[X]$. Since

$$\mathcal{O}_{\mathfrak{X},p} \cong \mathcal{O}_{\mathbb{C}^{n+1}\times\mathrm{Def}(X),(0,[X])}/(z_0^2+\cdots+z_n^2+f) \tag{2.2}$$

by e.g. [33, pp.103, (6.7)], this implies the smoothness of $\mathfrak{X}$ at p. $\square$

Let $\mathfrak{p}\colon (\mathfrak{X},X) \to (\mathrm{Def}(X),[X])$ be the Kuranishi family of X.

Proposition 2.8. *There exist a pointed projective variety $(B,0)$, a projective variety $\mathfrak{Z}$, and a surjective flat holomorphic map $f\colon \mathfrak{Z} \to B$ such that the deformation germ $f\colon (\mathfrak{Z}, f^{-1}(0)) \to (B,0)$ is isomorphic to $\mathfrak{p}\colon (\mathfrak{X},X) \to (\mathrm{Def}(X),[X])$. In particular, the map $\mathfrak{p}\colon \mathfrak{X} \to \mathrm{Def}(X)$ is projective.*

Proof. See [40, pp. 441, l.7-l.12]. $\square$

2.1.4. *The Serre duality for Calabi-Yau varieties with at most one ODP.* Let

$$\langle\cdot,\cdot\rangle\colon H^{n-1}(X,\Omega^1_X\otimes K_X)\times \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X\otimes K_X,K_X)\to H^n(X,K_X)\cong\mathbb{C}$$

be the Yoneda product. Since X is compact, the Yoneda product is a perfect pairing by [1, Th. 4.1 and Th. 4.2]. Hence we get by Proposition 2.5

$$H^{n-1}(X,\Omega^1_X\otimes K_X) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X\otimes K_X,K_X)^\vee = (T_{\mathrm{Def}(X),[X]})^\vee = \Omega^1_{\mathrm{Def}(X),[X]}.$$

If X is smooth, then $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X\otimes K_X,K_X) = H^1(X,\Theta_X)$ and the Yoneda product is given by the ordinary Serre duality pairing [1, Th. 4.2].

Let $H^{n-1}_c(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)$ be the cohomology with compact support.

Lemma 2.9. *The natural map $H^{n-1}_c(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)\to H^{n-1}(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)$ is an isomorphism. Under this isomorphism, the Yoneda product $\langle\cdot,\cdot\rangle$ coincides with the Serre duality pairing on the regular part of X:*

$$H^{n-1}_c(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)\times H^1(X_{\mathrm{reg}},\Theta_X)\to H^n_c(X_{\mathrm{reg}},\,K_X)\cong\mathbb{C}.$$

Proof. Since $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X) = \mathrm{Ext}^1_{\mathcal{O}_{X_{\mathrm{reg}}}}(\Omega^1_{X_{\mathrm{reg}}},\mathcal{O}_{X_{\mathrm{reg}}})$ by [54, Prop. 1.1], the Serre duality for open manifolds [1, Th. 4.1 and Th. 4.2] yields that

$$H^{n-1}(X,\Omega^1_X\otimes K_X) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)^\vee = \mathrm{Ext}^1_{\mathcal{O}_{X_{\mathrm{reg}}}}(\Omega^1_{X_{\mathrm{reg}}},\mathcal{O}_{X_{\mathrm{reg}}}) = H^{n-1}_c(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)$$

and that the Yoneda product pairing

$$H^{n-1}_c(X_{\mathrm{reg}},\Omega^1_X\otimes K_X)\times \mathrm{Ext}^1_{\mathcal{O}_{X_{\mathrm{reg}}}}(\Omega^1_{X_{\mathrm{reg}}}\otimes K_X,K_X)\to H^n_c(X_{\mathrm{reg}},\,K_X)$$

is perfect. Since X_{reg} is smooth, $\mathrm{Ext}^1_{\mathcal{O}_{X_{\mathrm{reg}}}}(\Omega^1_{X_{\mathrm{reg}}} \otimes K_X, K_X) = H^1(X_{\mathrm{reg}}, \Theta_X)$ and the Yoneda product pairing $\langle\cdot,\cdot\rangle$ coincides with the Serre duality pairing. $\square$

2.2. The locally-freeness of the direct image sheaves: the case $n = 3$

Let $n \geq 3$. Let X be a smoothable Calabi-Yau n-fold with at most one ODP. Let $\pi\colon (\mathcal{X}, X) \to (S, 0)$ be a smoothing of X. Set $\Omega^1_{\mathcal{X}/S} := \Omega^1_{\mathcal{X}}/\pi^*\Omega^1_S$.

Lemma 2.10. *The sheaf $\Omega^1_{\mathcal{X}/S}$ is a flat $\mathcal{O}_S$-module.*

Proof. Since $\Omega^1_{\mathcal{X}/S,\,x} \cong \mathcal{O}^{\oplus n}_{\mathcal{X},\,x}$ for $x \in \mathcal{X} \setminus \Sigma_\pi$, it suffices to prove the assertion for $x \in \Sigma_\pi$. Let $(\mathfrak{Y}, o) \to (\mathrm{Def}(A_1), 0) \cong (\mathbb{C}, 0)$ be the Kuranishi family of an ODP o. There exists a map $f\colon (S, \pi(x)) \to (\mathrm{Def}(A_1), 0)$ such that $(\mathcal{X}, x) \to (S, \pi(x))$ is induced from $(\mathfrak{Y}, o) \to (\mathrm{Def}(A_1), 0)$ by f. Let $p\colon \mathcal{X} = \mathfrak{Y} \times_{\mathrm{Def(A_1)}} S \to \mathfrak{Y}$ be the projection. Since $\Omega^1_{\mathcal{X}/S,\,x} = p^*\Omega^1_{\mathfrak{Y}/\mathrm{Def}(A_1)}$, the assertion follows from the fact that $\Omega^1_{\mathfrak{Y}/\mathrm{Def}(A_1),o}$ is a flat $\mathcal{O}_{\mathrm{Def}(A_1),0}$-module (cf. [41, p. 13, l. 28–p. 14, l. 1]). $\square$

Let us consider the case $S = \mathrm{Def}(X)$. Let $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X.

Theorem 2.11. *If $n = 3$, the function $\mathrm{Def}(X) \ni s \to h^q(X_s, \Omega^1_{X_s}) \in \mathbb{Z}$ is constant for all $q \geq 0$. In particular, $R^q\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is a locally free $\mathcal{O}_{\mathrm{Def}(X)}$-module on $\mathrm{Def}(X)$ for all $q \geq 0$.*

The proof of this theorem is divided into the four lemmas below.

Lemma 2.12. *If $n \geq 3$, the function $\mathrm{Def}(X) \ni s \to h^{n-1}(X_s, \Omega^1_{X_s}) \in \mathbb{Z}$ is constant. In particular, $R^{n-1}\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is a locally free $\mathcal{O}_{\mathrm{Def}(X)}$-module on $\mathrm{Def}(X)$.*

Proof. Since $K_X \cong \mathcal{O}_X$, we have

$$T_{\mathrm{Def}(X),[X]} \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \mathrm{Ext}^1_{\mathcal{O}_X}(K_X \otimes \Omega^1_X, K_X) = H^{n-1}(X, K_X \otimes \Omega^1_X)^\vee,$$

where the first isomorphism follows from Proposition 2.5, the second equality follows from the triviality of K_X, and the third equality follows from the Serre duality [24, Chap. III Th. 7.6 (b) (iii)]. Notice that we can apply the Serre duality to X, because X has at most one ODP and hence X is Cohen-Macaulay [24, Chap. II Th. 8.21, Prop. 8.23]. Since $K_X \cong \mathcal{O}_X$, we get $h^{n-1}(X, \Omega^1_X) = \dim T_{\mathrm{Def}(X),[X]}$. The smoothness of $\mathrm{Def}(X)$ at $[X]$ implies that the function on $\mathrm{Def}(X)$

$$\mathrm{Def}(X) \ni s \to \dim T_{\mathrm{Def}(X),s} = \dim T_{\mathrm{Def}(X_s),[X_s]} = h^{n-1}(X_s, \Omega^1_{X_s}) \in \mathbb{Z}$$

is constant, for the Zariski tangent space coincides with the usual tangent space for smooth varieties. Notice that the first equality $\dim T_{\mathrm{Def}(X),s} = \dim T_{\mathrm{Def}(X_s),[X_s]}$ follows from [16, Sect. 8.2]. Since $\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is a flat $\mathcal{O}_{\mathrm{Def}(X)}$-module by Lemmas 2.7 and 2.10, $R^{n-1}\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is locally free by [1, Chap. 3, Th. 4.12 (ii)]. $\square$

Lemma 2.13. *If $n = 3$, then $h^3(X_s, \Omega^1_{X_s}) = 0$ for all $s \in \mathrm{Def}(X)$. In particular, $R^3\pi_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)} = 0$.*

Proof. See [40, p. 432, l.23]. $\square$

Lemma 2.14. *If $n = 3$, the function $\mathrm{Def}(X) \ni s \to h^1(X_s, \Omega^1_{X_s}) \in \mathbb{Z}$ is constant. In particular, $R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is a locally free $\mathcal{O}_{\mathrm{Def}(X)}$-module.*

Proof. Since $\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is a flat $\mathcal{O}_{\mathrm{Def}(X)}$-module, the function $\mathrm{Def}(X) \ni s \to \chi(X_s, \Omega^1_{X_s}) \in \mathbb{Z}$ is constant, where $\chi(X_s, \Omega^1_{X_s})$ denotes the Euler characteristic of $\Omega^1_{X_s}$. Since $h^q(X_s, \Omega^1_{X_s})$ is independent of $s \in \mathrm{Def}(X)$ for all $q \neq 1$ by Lemmas 2.12 and 2.13, this implies that $h^1(X_s, \Omega^1_{X_s})$ is independent of $s \in \mathrm{Def}(X)$. □

Lemma 2.15. *If $n = 3$, then $R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}}$ is locally free. Moreover, the restriction map $R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}} \to R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ is an isomorphism of $\mathcal{O}_{\mathrm{Def}(X)}$-modules.*

Proof. Set $N := \dim \mathrm{Def}(X)$. The short exact sequence of sheaves on $\mathfrak{X}$

$$0 \to \mathcal{O}_{\mathfrak{X}}^{\oplus N} \cong \mathfrak{p}^*\Omega^1_{\mathrm{Def}(X)} \to \Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/\mathrm{Def}(X)} \to 0$$

induces the long exact sequence of direct images

$$\cdots \longrightarrow R^1\mathfrak{p}_*\mathfrak{p}^*\Omega^1_{\mathrm{Def}(X)} \longrightarrow R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}} \longrightarrow R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)} \longrightarrow R^2\mathfrak{p}_*\mathfrak{p}^*\Omega^1_{\mathrm{Def}(X)} \longrightarrow \cdots$$

Since $R^1\mathfrak{p}_*\mathfrak{p}^*\Omega^1_{\mathrm{Def}(X)} = (R^1\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}})^{\oplus N} = 0$ and $R^2\mathfrak{p}_*\mathfrak{p}^*\Omega^1_{\mathrm{Def}(X)} = (R^2\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}})^{\oplus N} = 0$ by Definition 2.1 (ii), the second assertion follows from the above exact sequence.

By the same argument as above, we see that the restriction map $H^1(X_s, \Omega^1_{\mathfrak{X}}|_{X_s}) \to H^1(X_s, \Omega^1_{X_s})$ is an isomorphism for all $s \in \mathrm{Def}(X)$. Hence $h^1(X_s, \Omega^1_{\mathfrak{X}}|_{X_s})$ is independent of $s \in \mathrm{Def}(X)$ by Lemma 2.14. This, together with [1, Chap. 3, Th. 4.12 (ii)] proves the first assertion. □

Theorem 2.11 follows from Lemmas 2.12, 2.13, 2.14, and 2.15. □

Let $H^2(X, \mathbb{Z})_{\mathrm{Def}(X)}$ be the constant sheaf on $\mathrm{Def}(X)$ with stalk $H^2(X, \mathbb{Z})$. By [40, Prop. 6.1], $R^2\mathfrak{p}_*\mathbb{Z}$ is isomorphic to the constant sheaf $H^2(X, \mathbb{Z})_{\mathrm{Def}(X)}$.

Since $R^1\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}} = R^2\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}} = 0$ by Definition 2.1 (ii), the exponential sequence on $\mathfrak{X}$ induces the exact sequence of direct images

$$0 = R^1\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}} \longrightarrow R^1\mathfrak{p}_*\mathcal{O}^*_{\mathfrak{X}} \xrightarrow{\cong} R^2\mathfrak{p}_*\mathbb{Z} \longrightarrow R^2\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}} = 0. \tag{2.3}$$

For a holomorphic line bundle $\mathcal{L} \in H^1(\mathfrak{X}, \mathcal{O}^*_{\mathfrak{X}})$, the Dolbeault cohomology class of the Chern form $c_1(\mathcal{L}, h) \in H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}})$ is independent of the choice of a Hermitian metric h on $\mathcal{L}$, which we will denote by $\mathfrak{C}_1(\mathcal{L})$. Since every element of $H^2(\mathfrak{X}, \mathbb{Z})$ is represented uniquely as the Chern class of an element of $H^1(\mathfrak{X}, \mathcal{O}^*_{\mathfrak{X}})$ by the isomorphism (2.3), we define the map $j\colon H^2(X, \mathbb{Z}) \to H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}})$ by

$$j\left(c_1(\mathcal{L})|_X\right) := \mathfrak{C}_1(\mathcal{L}), \qquad \mathcal{L} \in H^1(\mathfrak{X}, \mathcal{O}^*_{\mathfrak{X}}).$$

We regard $\mathfrak{C}_1(\mathcal{L})$ as an element of $H^0(\mathrm{Def}(X), R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)})$ after Lemma 2.15. Since $H^2(X, \mathbb{Z})$ is finitely generated, the map j extends to a homomorphism of $\mathcal{O}_{\mathrm{Def}(X)}$-modules

$$j\colon H^2(X, \mathbb{Z})_{\mathrm{Def}(X)} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Def}(X)} \to R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}.$$

Lemma 2.16. *The homomorphism j is an isomorphism of $\mathcal{O}_{\mathrm{Def}(X)}$-modules.*

Proof. Since $H^2(X, \mathbb{Z})_{\mathrm{Def}(X)} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Def}(X)}$ and $R^1\mathfrak{p}_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ are locally free by Lemma 2.15, it suffices to prove that $j|_X\colon H^2(X, \mathbb{C}) \to H^1(X, \Omega^1_X)$ is an isomorphism. Since $h^2(X, \mathbb{C}) = h^2(X_s, \mathbb{C})$ by [40, Prop. 6.1] and since $h^1(X_s, \Omega^1_{X_s}) = h^1(X, \Omega^1_X)$ by Lemma 2.14, we get $h^2(X, \mathbb{C}) = h^1(X, \Omega^1_X)$. Since $j|_X$ is surjective by [40, Lemma 2.2], it is an isomorphism. □

3. **Quillen metrics**

Throughout Section 3, we fix the following notation: Let X be a complex manifold. Let (F, h_F) be a holomorphic Hermitian vector bundle on X, which we also write $\overline{F} = (F, h_F)$ for simplicity.

3.1. **Analytic torsion and BCOV torsion**

In Subsection 3.1, assume that X is a compact Kähler manifold with Kähler metric g_X and with Kähler form γ_X. We set $\overline{X} = (X, g_X)$. Define $\overline{\Omega}^p_X$ to be the holomorphic vector bundle Ω^p_X equipped with the Hermitian metric induced from g_X.

Let $A^{p,q}_X(F)$ be the vector space of F-valued smooth (p,q)-forms on X. Set $S_F = \bigoplus_{q\geq 0} A^{0,q}_X(F)$. Let $\langle\cdot,\cdot\rangle$ be the Hermitian metric on $(\bigwedge T^{*(0,1)}X)\otimes F$ induced from g_X and h_F. The volume form of $\overline{X}$ is defined by $dv_X = \gamma_X^{\dim X}/(\dim X)!$. The L^2*-metric* is the Hermitian metric on S_F defined by

$$(s,s')_{L^2} := \frac{1}{(2\pi)^{\dim X}} \int_X \langle s(x), s'(x)\rangle_x \, dv_X(x), \qquad s, s' \in S_F.$$

Let $\bar\partial_F$ be the Dolbeault operator acting on S_F and let $\bar\partial^*_F$ be the formal adjoint of $\bar\partial_F$ with respect to $(\cdot,\cdot)_{L^2}$. Then $\square_F = (\bar\partial_F + \bar\partial^*_F)^2$ is the corresponding $\bar\partial$-Laplacian. Let $\sigma(\square_F)$ be the spectrum of $\square_F$ and let $E_F(\lambda)$ be the eigenspace of $\square_F$ with respect to the eigenvalue λ.

Let N and ϵ be the operators on S_F defined by $N = q$ and $\epsilon = (-1)^q$ on $A^{0,q}_X(F)$. Then N and ϵ preserve $E_F(\lambda)$.

The zeta function

$$\zeta_{\overline{F}}(s) := \sum_{\lambda\in\sigma(\square_F)\setminus\{0\}} \lambda^{-s}\,\mathrm{Tr}\,[\epsilon N|_{E_F(\lambda)}].$$

converges absolutely for $s\in\mathbb{C}$ with $\mathrm{Re}\, s \gg 1$. By [11, II, Th. 2.16, (2.98)], $\zeta_{\overline{F}}(s)$ has a meromorphic continuation to the complex plane, which is holomorphic at $s = 0$.

Definition 3.1. (i) The *analytic torsion* of $(\overline{X}, \overline{F})$ is defined by

$$\tau(\overline{X}, \overline{F}) := \exp(-\zeta'_{\overline{F}}(0)).$$

(ii) The *BCOV torsion* of $\overline{X}$ is defined by

$$\mathcal{T}_{\mathrm{BCOV}}(\overline{X}) := \prod_{p\geq 0} \tau(\overline{X}, \overline{\Omega}^p_X)^{(-1)^p p} = \exp[-\sum_{p\geq 0}(-1)^p p\, \zeta'_{\overline{\Omega}^p_X}(0)].$$

We refer the reader to [11], [46] for more details about analytic torsion.

3.2. **Quillen metrics**

Definition 3.2. (i) The *determinant of the cohomologies of* F is the complex line defined by

$$\lambda(F) := \bigotimes_{q\geq 0} (\det H^q(X,F))^{(-1)^q}.$$

(ii) The *BCOV line* is the complex line $\lambda(\Omega^\bullet_X)$ defined by

$$\lambda(\Omega^\bullet_X) := \bigotimes_{p\geq 0} \lambda(\Omega^p_X)^{(-1)^p p} = \bigotimes_{p,q\geq 0} (\det H^q(X,\Omega^p_X))^{(-1)^{p+q}p}.$$

Set $K^q(\overline{X},\overline{F}) = \ker\square_F \cap A_X^{0,q}(F)$. Then $K^q(\overline{X},\overline{F})$ inherits a metric from $(\cdot,\cdot)_{L^2}$. By Hodge theory, we have an isomorphism $H^q(X,F) \cong K^q(\overline{X},\overline{F})$. We define $h_{H^q(X,F)}$ to be the metric on $H^q(X,F)$ induced from the L^2-metric on $K^q(\overline{X},\overline{F})$ by this isomorphism.

Let $\|\cdot\|_{L^2,\lambda(F)}$ be the Hermitian metric on $\lambda(F)$ induced from $\{h_{H^q(X,F)}\}_{q\geq 0}$.

Definition 3.3. (i) The *Quillen metric* on $\lambda(F)$ is defined by

$$\|\alpha\|^2_{Q,\lambda(F)} := \tau(\overline{X},\overline{F}) \cdot \|\alpha\|^2_{L^2,\lambda(F)}, \qquad \alpha \in \lambda(F).$$

(ii) The *Quillen metric* on $\lambda(\Omega_X^\bullet)$ is defined by

$$\|\cdot\|^2_{Q,\lambda(\Omega_X^\bullet)} := \bigotimes_{p\geq 0} \|\cdot\|^{(-1)^p 2p}_{Q,\lambda(\Omega_X^p)} = \mathcal{T}_{\mathrm{BCOV}}(\overline{X}) \cdot \bigotimes_{p\geq 0} \|\cdot\|^{(-1)^p 2p}_{L^2,\lambda(\Omega_X^p)}.$$

Let $(F_1,h_{F_1}),\cdots,(F_l,h_{F_l})$ be holomorphic Hermitian vector bundles on X, and let $\|\cdot\|^2_{Q,\lambda(F_k)}$ be the Quillen metric on $\lambda(F_k)$. For $\otimes_{k=1}^l \alpha_k \in \bigotimes_{k=1}^l \lambda(F_k)$, we set $\|\otimes_{k=1}^l \alpha_k\|^2_{Q,\otimes_k\lambda(F_k)} := \prod_{k=1}^l \|\alpha_k\|^2_{Q,\lambda(F_k)}$. When the line $\lambda(F)$ is clear from the context, we write $\|\cdot\|_Q$ for $\|\cdot\|_{Q,\lambda(F)}$. We refer the reader to [11], [12], [44], [50] for more details about Quillen metrics.

3.3. The Serre duality

Let $n := \dim X$. By the Serre duality, the following pairing on the Dolbeault cohomology groups is perfect:

$$H^q(X,\Omega_X^p) \times H^{n-q}(X,\Omega_X^{n-p}) \ni (\alpha,\beta) \to \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_X \alpha\wedge\beta \in \mathbb{C}.$$

Let $\{\psi_i\}$ be an arbitrary basis of $H^q(X,\Omega_X^p)$, and let $\{\psi_i^\vee\}$ the dual basis of $H^{n-q}(X,\Omega_X^{n-p})$ with respect to the Serre duality pairing. Then the element of $\det H^p(X,\Omega_X^q) \otimes \det H^{n-p}(X,\Omega_X^{n-q})$ defined by

$$\mathbf{1}_{(p,q),(n-p,n-q)} := \bigwedge_i \psi_i \otimes \bigwedge_i \psi_i^\vee \tag{3.1}$$

is independent of the choice of a basis $\{\psi_i\}$ and is called the *canonical element*. Similarly, the following element of $\lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}$ is also called the canonical element:

$$\mathbf{1}_{p,n-p} = \mathbf{1}_{p,n-p}(X) := \bigotimes_{q=0}^n \mathbf{1}_{(p,q),(n-p,n-q)} \in \lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}.$$

Then $\mathbf{1}_{(p,q),(n-p,n-q)} = \mathbf{1}^{-1}_{(p,q),(n-p,n-q)}$ by (3.1).

Let $1_\mathbb{C}$ be the trivial Hermitian structure on $\mathbb{C}$, i.e., $1_\mathbb{C}(a) = |a|^2$ for $a\in\mathbb{C}$.

Proposition 3.4. *The following identity holds:*

$$\|\mathbf{1}_{p,n-p}\|_{L^2} = \|\mathbf{1}_{p,n-p}\|_Q = 1. \tag{3.2}$$

In particular, the canonical element $\mathbf{1}_{p,n-p}$ induces the following canonical isometries of the Hermitian lines:

$$\left(\lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}, \|\cdot\|_{L^2,\lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}}\right) \cong (\mathbb{C},1_\mathbb{C}), \tag{3.3}$$

$$\left(\lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}, \|\cdot\|_{Q,\lambda(\Omega_X^p)\otimes\lambda(\Omega_X^{n-p})^{(-1)^n}}\right) \cong (\mathbb{C},1_\mathbb{C}). \tag{3.4}$$

Proof. Let $\{\phi_i\}$ be a unitary basis of $H^q(X, \Omega_X^p)$ with respect to the L^2-metric. The dual basis of $\{\phi_i\}$ with respect to the Serre duality pairing is given by $\{\bar{*}\phi_i\}$, where $*\colon A_X^{p,q} \to A_X^{n-q,n-p}$ is the Hodge $*$-operator with respect to the metric g_X. By setting $\psi_i = \phi_i$ in (3.1), we get the first equality

$$\|\mathbf{1}_{(p,q),(n-p,n-q)}\|_{L^2} = 1, \tag{3.5}$$

which yields the isometry (3.3).

Let $\zeta_{p,q}(s)$ be the spectral zeta function of the $\bar{\partial}$-Laplacian acting on $A_X^{p,q}$. Since $\bar{*}^{-1}\square_{p,q}\bar{*} = \square_{n-p,n-q}$, we have $\zeta_{p,q}(s) = \zeta_{n-p,n-q}(s)$, which yields that

$$\tau(\overline{X}, \overline{\Omega}_X^p) = \tau(\overline{X}, \overline{\Omega}_X^{n-p})^{(-1)^{n+1}}. \tag{3.6}$$

The second isometry (3.4) follows from (3.3) and (3.6). $\square$

For more details about the Serre duality for Quillen metrics, we refer to [21, (9)].

3.4. **Characteristic classes**

In Subsections 3.4 and 3.5, we do *not* assume that X is compact Kähler.

3.4.1. *Chern forms.* For a square matrix A, set $\mathrm{Td}(A) := \det\left(\frac{A}{I-\exp(-A)}\right)$ and $\mathrm{ch}(A) := \mathrm{Tr}[e^A]$. Let $R(\overline{F})$ be the curvature of $\overline{F} = (F, h_F)$ with respect to the holomorphic Hermitian connection. The real closed forms on X defined by

$$\mathrm{Td}(F, h_F) := \mathrm{Td}\left(-\frac{1}{2\pi\sqrt{-1}}R(\overline{F})\right), \qquad \mathrm{ch}(F, h_F) := \mathrm{ch}\left(-\frac{1}{2\pi\sqrt{-1}}R(\overline{F})\right)$$

are called the *Todd form* and the *Chern character form* of $\overline{F}$, respectively.

Let $c_i(F, h_F)$ be the i-th Chern form of (F, h_F).

3.4.2. *Bott-Chern classes.* Let $\mathcal{E} : 0 \to E_0 \to E_1 \to \cdots \to E_m \to 0$ be an exact sequence of holomorphic vector bundles on X, equipped with Hermitian metrics h_i, $i = 0, \ldots, m$. We set $\overline{\mathcal{E}} := (\mathcal{E}, \{h_i\}_{i=0}^m)$. By [11, I, Th. 1.29 and Eqs. (0.5), (1.124)], one has the Bott-Chern secondary class $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \bigoplus_{p\geq 0} A^{p,p}(X)/\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$ associated to the Chern character and $\overline{\mathcal{E}}$ such that

$$dd^c\,\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \sum_{i=0}^{m}(-1)^{i+1}\mathrm{ch}(E_i, h_i).$$

Consider the case where $m = 1$ and $E_0 = E_1 = E$. Let h' and h be Hermitian metrics of E_0 and E_1, respectively. By [11, I, Th. 1.27] or [20, Sect. 1.2.4], one has the Bott-Chern secondary class $\widetilde{\mathrm{ch}}(E;\, h, h') \in \bigoplus_{p\geq 0} A^{p,p}(X)/\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$ such that

$$dd^c\widetilde{\mathrm{ch}}(E;\, h, h') = \mathrm{ch}(E, h) - \mathrm{ch}(E, h').$$

When $\mathrm{rk}(E) = 1$, we have the following explicit formula by [20, I, (1.2.5.1), (1.3.1.2)]:

$$\widetilde{\mathrm{ch}}(E;\, h, h') = \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{a+b=m-1} c_1(E, h)^a c_1(E, h')^b \log\left(\frac{h'}{h}\right). \tag{3.7}$$

Similarly, let $\widetilde{\mathrm{Td}}(E; h, h') \in \bigoplus_{p\geq 0} A^{p,p}(X)/\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$ denote the Bott-Chern secondary class associated to the Todd form such that

$$dd^c\,\widetilde{\mathrm{Td}}(E; h, h') = \mathrm{Td}(E, h) - \mathrm{Td}(E, h').$$

For more details about Bott-Chern classes, we refer to [11], [20], [50].

3.5. The curvature formulas

Let S be a complex manifold and let $\pi\colon X \to S$ be a proper surjective holomorphic submersion. Then every fiber of π is a compact complex manifold. The map $\pi\colon X \to S$ is said to be *locally Kähler* if for every $s \in S$ there is an open subset $U \ni s$ such that $\pi^{-1}(U)$ possesses a Kähler metric. We set $X_s = \pi^{-1}(s)$ for $s \in S$.

Let $TX/S := \ker \pi_* \subset TX$ be the relative holomorphic tangent bundle of the family $\pi\colon X \to S$. Set $\Omega^p_{X/S} := \bigwedge^p (TX/S)^\vee$ and $K_{X/S} := K_X \otimes (\pi^* K_S)^{-1} = \Omega^{\dim X - \dim S}_{X/S}$.

A C^∞ Hermitian metric on TX/S is said to be *fiberwise Kähler* if the induced metric on X_s is Kähler for all $s \in S$. By Kodaira-Spencer, there exists an fiberwise Kähler metric on TX/S if and only if every X_s possesses a Kähler metric.

Assume that every fiber X_s possesses a Kähler metric. Let $g_{X/S}$ be a fiberwise Kähler metric on TX/S. Set $g_s = g_{X/S}|_{X_s}$ and $\overline{X}_s = (X_s, g_s)$ for $s \in S$. We define $\overline{\Omega}^p_{X_s}$ to be the holomorphic vector bundle $\Omega^p_{X_s}$ equipped with the Hermitian metric induced from g_s. When $p = 0$, $\overline{\Omega}^0_{X_s}$ is defined as the trivial line bundle $\mathcal{O}_{X_s}$ equipped with the trivial Hermitian metric.

Since $\dim H^q(X_s, \Omega^p_{X_s})$ is locally constant, the direct image sheaf $R^q\pi_*\Omega^p_{X/S}$ is locally free for all $p, q \geq 0$ and is identified with the corresponding holomorphic vector bundle over S. Set

$$\lambda(\Omega^\bullet_{X/S}) := \bigotimes_{p,q\geq 0} (\det R^q\pi_*\Omega^p_{X/S})^{(-1)^{p+q}p}.$$

Via the natural fiberwise identification $\lambda(\Omega^\bullet_{X/S})|_s = \lambda(\Omega^\bullet_{X_s})$ for all $s \in S$, $\lambda(\Omega^\bullet_{X/S})$ is equipped with the Hermitian metric $\|\cdot\|_{\lambda(\Omega^\bullet_{X/S}),Q}$ defined by

$$\|\cdot\|_{Q,\lambda(\Omega^\bullet_{X/S})}(s) := \|\cdot\|_{Q,\lambda(\Omega^\bullet_{X_s})}, \qquad s \in S,$$

which is smooth by [11, III, Cor. 3.9]. We set $\lambda(\Omega^\bullet_{X/S})_Q := (\lambda(\Omega^\bullet_{X/S}), \|\cdot\|_{Q,\lambda(\Omega^\bullet_{X/S})})$.

Since $\dim K^q(\overline{X}_s, \overline{\Omega}^p_{X_s})$ is locally constant, there exists a C^∞ vector bundle $\mathcal{K}^{p,q}(X/S)$ over S such that $\mathcal{K}^{p,q}(X/S)_s = K^q(\overline{X}_s, \overline{\Omega}^p_{X_s})$ for all $s \in S$. Then the fiberwise isomorphism $H^q(X_s, \Omega^p_{X_s}) \cong K^q(\overline{X}_s, \overline{\Omega}^p_{X_s})$ via Hodge theory induces an isomorphism of C^∞ vector bundles $R^q\pi_*\Omega^p_{X/S} \cong \mathcal{K}^{p,q}(X/S)$. Let $h_{R^q\pi_*\Omega^p_{X/S}}$ be the C^∞ Hermitian metric on $R^q\pi_*\Omega^p_{X/S}$ induced from the L^2-metric on $\mathcal{K}^{p,q}(X/S)$ by this isomorphism. We define $\overline{R^q\pi_*\Omega^p_{X/S}} := (R^q\pi_*\Omega^p_{X/S}, h_{R^q\pi_*\Omega^p_{X/S}})$.

Let $\mathcal{T}_{\mathrm{BCOV}}(X/S)$ be the function on S defined by

$$\mathcal{T}_{\mathrm{BCOV}}(X/S)(s) := \mathcal{T}_{\mathrm{BCOV}}(\overline{X}_s) = \prod_{p\geq 0} \tau(\overline{X}_s, \overline{\Omega}^p_{X_s})^{(-1)^p p}, \qquad s \in S.$$

For a differential form φ, $[\varphi]^{(p,q)}$ denotes the component of bidegree (p,q) of φ.

Theorem 3.5. *Assume that the map $\pi\colon X \to S$ is locally Kähler and set $n = \dim X - \dim S$. Then $\mathcal{T}_{\mathrm{BCOV}}(X/S)$ lies in $C^\infty(S)$, and the following equation of C^∞ $(1,1)$-forms on S holds:*

$$\begin{aligned} c_1(\lambda(\Omega^\bullet_{X/S})_Q) &= -dd^c \log \mathcal{T}_{\mathrm{BCOV}}(X/S) + \sum_{q\geq 0} (-1)^{p+q} p\, c_1(\overline{R^q\pi_*\Omega^p_{X/S}}) \\ &= -\frac{1}{12}\pi_* \left[c_1(TX/S, g_{X/S})\, c_n(TX/S, g_{X/S})\right]^{(1,1)}. \end{aligned}$$

Proof. See [7, pp. 374] and [11, Th. 0.1]. □

4. The BCOV invariant of Calabi-Yau manifolds

Throughout Section 4, we fix the following notation: Let X be a *smooth* Calabi-Yau n-fold. Let $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X.

Let g be a Kähler metric on X with Kähler form γ. We define $\mathrm{Vol}(X,\gamma) := (2\pi)^{-n}\int_X \gamma^n/n! = \|1\|_{L^2}^2$. Notice that our definition of $\mathrm{Vol}(X,\gamma)$ is different from the ordinary one because of the factor $(2\pi)^{-n}$. We set $c_i(X,\gamma) := c_i(TX, g)$ and $\chi(X) := \int_X c_n(X,\gamma)$. Let $\eta \in H^0(X, \Omega_X^n) \setminus \{0\}$.

4.1. The BCOV Hermitian line

Recall that the L^2-norm on $H^0(X,\Omega_X^n)$ is independent of the choice of a Kähler metric g because

$$\|\eta\|_{L^2}^2 = (2\pi)^{-n}(\sqrt{-1})^{n^2}\int_X \eta\wedge\bar{\eta}.$$

After [60, Sect. 5.1], we make the following:

Definition 4.1. (i) For $\overline{X} = (X,\gamma)$, define $\mathcal{A}(\overline{X}) = \mathcal{A}(X,\gamma) \in \mathbb{R}$ by

$$\mathcal{A}(\overline{X}) := \mathrm{Vol}(X,\gamma)^{\frac{\chi(X)}{12}} \exp\left[-\frac{1}{12}\int_X \log\left(\frac{(\sqrt{-1})^{n^2}\eta\wedge\bar{\eta}}{\gamma^n/n!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|_{L^2}^2}\right) c_n(X,\gamma)\right].$$

(ii) The *BCOV metric* is the Hermitian structure $\|\cdot\|_{\lambda(\Omega_X^\bullet)}$ on $\lambda(\Omega_X^\bullet)$ defined by

$$\|\cdot\|_{\lambda(\Omega_X^\bullet)}^2 := \mathcal{A}(\overline{X})\cdot\|\cdot\|_{Q,\lambda(\Omega_X^\bullet)}^2.$$

(iii) The *BCOV Hermitian line* is defined by

$$\overline{\lambda(\Omega_X^\bullet)} := (\lambda(\Omega_X^\bullet), \|\cdot\|_{\lambda(\Omega_X^\bullet)}).$$

Remark 4.2. By Yau [58], every Kähler class on X contains a unique Ricci-flat Kähler form. If κ is a Ricci-flat Kähler form on X, then

$$\frac{\kappa^n/n!}{(\sqrt{-1})^{n^2}\eta\wedge\bar{\eta}} = \frac{\mathrm{Vol}(X,\kappa)}{\|\eta\|_{L^2}^2},$$

and hence $\log\mathcal{A}(X,\kappa) = \frac{\chi(X)}{12}\log\mathrm{Vol}(X,\kappa)$ in this case.

4.2. The Weil-Petersson metric and the Hodge metric

To compute the curvature of the BCOV Hermitian line bundles, let us recall the definitions of the Weil-Petersson metric [53] and the Hodge metric [35], [36].

By Proposition 2.5, the homomorphism of $\mathcal{O}_{\mathrm{Def}(X)}$-modules on $\mathrm{Def}(X)$ induced by the Kodaira-Spencer map

$$\rho_{\mathrm{Def}(X)}\colon \Theta_{\mathrm{Def}(X)} \to R^1\mathfrak{p}_*\Theta_{\mathfrak{X}/\mathrm{Def}(X)}$$

is an isomorphism, which is called the *Kodaira-Spencer isomorphism* in this paper.

Since $H^{n-1}(X_s, \Omega_{X_s}^1) \subset H^n(X_s,\mathbb{C})$ consists of primitive cohomology classes for all $s \in \mathrm{Def}(X)$, the L^2-metric on $R^1\mathfrak{p}_*\Omega_{\mathfrak{X}/\mathrm{Def}(X)}^{n-1}$ is independent of the choice of a fiberwise-Kähler metric on $T\mathfrak{X}/\mathrm{Def}(X)$ by e.g. [57, Th. 6.32]. We will often denote the L^2-metric $h_{R^1\mathfrak{p}_*\Omega_{\mathfrak{X}/\mathrm{Def}(X)}^{n-1}}$ on $R^1\mathfrak{p}_*\Omega_{\mathfrak{X}/\mathrm{Def}(X)}^{n-1}$ by $(\cdot,\cdot)_{L^2}$. Then

$$(\xi,\zeta)_{L^2} = -(2\pi)^{-n}(\sqrt{-1})^{n^2}\int_X \xi\wedge\overline{\zeta}, \qquad \xi,\zeta \in H^1(X,\Omega_X^{n-1}).$$

For $s \in \mathrm{Def}(X)$, let $\rho_s \colon T_{\mathrm{Def}(X),s} \to H^1(X_s, \Theta_{X_s})$ be the Kodaira-Spencer map, and let $\eta_s \in H^0(X_s, \Omega^n_{X_s}) \setminus \{0\}$. Let $\iota(\cdot)$ be the interior product.

Definition 4.3. The *Weil-Petersson metric* g_{WP} on $\mathrm{Def}(X)$ is defined by

$$g_{\mathrm{WP}}(u,v) := -\frac{\int_{X_s} \iota(\rho_s(u))\eta_s \wedge \overline{\iota(\rho_s(v))\eta_s}}{\int_{X_s} \eta_s \wedge \overline{\eta}_s} = \frac{(\iota(\rho_s(u))\eta_s, \iota(\rho_s(v))\eta_s)_{L^2}}{\|\eta_s\|^2_{L^2}}$$

for $u, v \in T_{\mathrm{Def}(X),s}$. Let ω_{WP} be the Kähler form of g_{WP}.

Let $\eta_{\mathfrak{X}/\mathrm{Def}(X)}$ be a local basis of $\mathfrak{p}_* K_{\mathfrak{X}/\mathrm{Def}(X)}$. By e.g. [53, Th. 2], we have

$$\omega_{\mathrm{WP}} = -dd^c \log \|\eta_{\mathfrak{X}/\mathrm{Def}(X)}\|^2_{L^2} = c_1(\mathfrak{p}_* K_{\mathfrak{X}/\mathrm{Def}(X)}, \|\cdot\|_{L^2}). \tag{4.1}$$

Proposition 4.4. *The Kodaira-Spencer map $\rho_{\mathrm{Def}(X)}$ induces an isometry of the following holomorphic Hermitian vector bundles on $\mathrm{Def}(X)$:*

$$(\Theta_{\mathrm{Def}(X)}, g_{\mathrm{WP}}) \otimes (\mathfrak{p}_* K_{\mathfrak{X}/\mathrm{Def}(X)}, \|\cdot\|_{L^2}) \cong (R^1\mathfrak{p}_*\Omega^{n-1}_{\mathfrak{X}/\mathrm{Def}(X)}, h_{R^1\mathfrak{p}_*\Omega^{n-1}_{\mathfrak{X}/\mathrm{Def}(X)}}).$$

In particular, $\rho_{\mathrm{Def}(X)}$ induces an isometry of the following holomorphic Hermitian line bundles on $\mathrm{Def}(X)$:

$$\begin{aligned}&(\det R^1\mathfrak{p}_*\Omega^{n-1}_{\mathfrak{X}/\mathrm{Def}(X)}, \det h_{R^1\mathfrak{p}_*\Omega^{n-1}_{\mathfrak{X}/\mathrm{Def}(X)}})\\ &\cong (\det \Theta_{\mathrm{Def}(X)}, \det g_{\mathrm{WP}}) \otimes (\mathfrak{p}_* K_{\mathfrak{X}/\mathrm{Def}(X)}, \|\cdot\|_{L^2})^{\otimes h^{1,n-1}(X)}.\end{aligned}$$

Proof. The Kodaira-Spencer isomorphism is given by

$$\Theta_{\mathrm{Def}(X)} \otimes \mathfrak{p}_* K_{\mathfrak{X}/\mathrm{Def}(X)} \ni u \otimes \eta \to \iota(\rho_{\mathrm{Def}(X)}(u))\eta \in R^1\mathfrak{p}_*\Omega^{n-1}_{\mathfrak{X}/\mathrm{Def}(X)}.$$

Hence $(\iota(\rho_{\mathrm{Def}(X)}(u))\,\eta, \iota(\rho_{\mathrm{Def}(X)}(v))\,\eta)_{L^2} = g_{\mathrm{WP}}(u,v) \cdot \|\eta\|^2_{L^2}$ by Definition 4.3. □

Definition 4.5. The Ricci form of the Weil-Petersson metric is the Chern form of the Hermitian line bundle $(\det \Theta_{\mathrm{Def}(X)}, \det g_{\mathrm{WP}})$:

$$\mathrm{Ric}\,\omega_{\mathrm{WP}} := c_1(\det \Theta_{\mathrm{Def}(X)}, \det g_{\mathrm{WP}}).$$

Proposition 4.6. *The following identities hold:*

$$c_1(\det R^{n-p}\pi_*\Omega^p_{\mathfrak{X}/\mathrm{Def}(X)}, \|\cdot\|_{L^2}) = \begin{cases} -\omega_{\mathrm{WP}} & (p=0) \\ -\mathrm{Ric}\,\omega_{\mathrm{WP}} - h^{1,n-1}(X)\,\omega_{\mathrm{WP}} & (p=1) \\ \mathrm{Ric}\,\omega_{\mathrm{WP}} + h^{1,n-1}(X)\,\omega_{\mathrm{WP}} & (p=n-1) \\ \omega_{\mathrm{WP}} & (p=n). \end{cases}$$

Proof. The assertion for $p = 0, n$ follows from (4.1). The assertion for $p = 1, n-1$ follows from Proposition 4.4 and the Serre duality. □

See [17, Sect. 2] for a generalization of Proposition 4.6. In the case $n = 3$, the following positivity result for $\mathrm{Ric}\,\omega_{\mathrm{WP}} + (h^{1,2}(X)+3)\,\omega_{\mathrm{WP}}$ shall be crucial in Sect. 7.

Proposition 4.7. *When $n = 3$, the $(1,1)$-form $\mathrm{Ric}\,\omega_{\mathrm{WP}} + (h^{1,2}(X)+3)\,\omega_{\mathrm{WP}}$ is a Kähler form on $\mathrm{Def}(X)$.*

Proof. See [36, Th. 1.1]. □

Definition 4.8. When $n = 3$, the *Hodge form* on $\mathrm{Def}(X)$ is the positive $(1,1)$-form on $\mathrm{Def}(X)$ defined as

$$\omega_{\mathrm{H}} := \mathrm{Ric}\,\omega_{\mathrm{WP}} + (h^{1,2}(X)+3)\,\omega_{\mathrm{WP}}.$$

The corresponding Kähler metric on the Kuranishi space $\mathrm{Def}(X)$ is called the *Hodge metric* on $\mathrm{Def}(X)$.

The Hodge metric is related to the invariant Hermitian metric on the period domain for Calabi-Yau threefolds as follows. Let X be a polarized smooth Calabi-Yau threefold. Let D be the classifying space for the polarized Hodge structures of weight 3 on $H^3(X,\mathbb{Z})/\text{Torsion}$ defined by Griffiths e.g. [23, Sect. 2]. Let F^i $(i=1,2,3)$ be the Hodge bundles on D. Let ω_D be the invariant Hermitian metric of D. Let $f\colon \mathrm{Def}(X)\to D$ be the period map. Then we have
(a) $\omega_{\mathrm{WP}} = f^*(c_1(F^3,\|\cdot\|_{L^2}))$ [57];
(b) Up to a constant, $\omega_{\mathrm{H}} = f^*(\omega_D)$ [35]. In particular, ω_{H} is always Kählerian.

We refer to e.g. [23] for more details about the classifying space D.

4.3. The curvature formula for the BCOV Hermitian line bundles

Let $\pi\colon (\mathcal{X},X)\to(S,0)$ be a flat deformation of X. Set $X_s=\pi^{-1}(s)$ for $s\in S$. Let $g_{\mathcal{X}/S}$ be a fiberwise-Kähler metric on $T\mathcal{X}/S$. Then the line bundle $\lambda(\Omega^\bullet_{\mathcal{X}/S})$ on S is equipped with the BCOV metric $\|\cdot\|_{\lambda(\Omega^\bullet_{\mathcal{X}/S})}$ with respect to $g_{\mathcal{X}/S}$.

Let $\mu\colon (S,0)\to(\mathrm{Def}(X),[X])$ be the holomorphic map such that the family $\pi\colon(\mathcal{X},X)\to(S,0)$ is induced from the Kuranishi family by μ. Then we have

$$c_1(\pi_*\omega_{\mathcal{X}/S},\|\cdot\|_{L^2}) = \mu^*\omega_{\mathrm{WP}}$$

near $s=0$. Let $\eta_{\mathcal{X}/S}$ be a local basis of $\pi_*\omega_{\mathcal{X}/S}$ and set

$$\omega_{\mathrm{WP},\,\mathcal{X}/S} := \mu^*\omega_{\mathrm{WP}} = -dd^c\log\|\eta_{\mathcal{X}/S}\|^2_{L^2} = c_1(\pi_*\omega_{\mathcal{X}/S},\|\cdot\|_{L^2}).$$

Theorem 4.9. *The following identity of* $(1,1)$*-forms on* $(S,0)$ *holds:*

$$c_1(\overline{\lambda(\Omega^\bullet_{\mathcal{X}/S})}) = \frac{\chi(X)}{12}\,\omega_{\mathrm{WP},\,\mathcal{X}/S}.$$

Proof. We follow [60, Sect. 5.2]. Since the assertion is of local nature, it suffices to prove it when $S\cong\Delta^{\dim S}$. Then $\pi_*K_{\mathcal{X}/S}\cong\mathcal{O}_S$. Let $\eta_{\mathcal{X}/S}\in H^0(S,\pi_*K_{\mathcal{X}/S})$ be a nowhere vanishing holomorphic section. For $s\in S$, set $\eta_s=\eta_{\mathcal{X}/S}|_{X_s}$. Then $\eta_s\in H^0(X_s,K_{X_s})\setminus\{0\}$ and $\eta_{\mathcal{X}/S}$ are identified with the family of holomorphic n-forms $\{\eta_s\}_{s\in S}$ varying holomorphically in $s\in S$. Define $\|\eta_{\mathcal{X}/S}\|^2_{L^2}\in C^\infty(S)$ by

$$\|\eta_{\mathcal{X}/S}\|^2_{L^2}(s) = \|\eta_s\|^2_{L^2},\qquad s\in S.$$

Set $g_s=g_{\mathcal{X}/S}|_{X_s}$. Then $g_{\mathcal{X}/S}$ is identified with the family of Kähler metrics $\{g_s\}_{s\in S}$. Let γ_s be the Kähler form of h_s. Let $\gamma_{\mathcal{X}/S}=\{\gamma_s\}_{s\in S}$ be the family of Kähler forms associated to $g_{\mathcal{X}/S}$.

Define the C^∞ functions $\mathrm{Vol}(\mathcal{X}/S)$ and $\mathcal{A}(\mathcal{X}/S)$ on S by

$$\mathrm{Vol}(\mathcal{X}/S)(s)=\mathrm{Vol}(X_s,\gamma_s),\qquad A(\mathcal{X}/S)(s)=A(X_s,\gamma_s),\qquad s\in S.$$

Let $c_i(\mathcal{X}/S)$ be the i-th Chern form of the holomorphic Hermitian vector bundle $(T\mathcal{X}/S,g_{\mathcal{X}/S})$. Since

$$c_1(\mathcal{X}/S) = -c_1(K_{\mathcal{X}/S},\det g^{-1}_{\mathcal{X}/S}) = dd^c\log\left(\frac{(\sqrt{-1})^{n^2}\eta_{\mathcal{X}/S}\wedge\overline{\eta}_{\mathcal{X}/S}}{\gamma^n_{\mathcal{X}/S}/n!}\right),$$

the following identity of $(1,1)$-forms on $\mathcal{X}$ holds:

$$\begin{aligned}c_1(\mathcal{X}/S) &= -\pi^*\left\{\omega_{\mathrm{WP},\mathcal{X}/S}+dd^c\log\mathrm{Vol}(\mathcal{X}/S)\right\}\\ &\quad + dd^c\log\left\{\frac{(\sqrt{-1})^{n^2}\eta_{\mathcal{X}/S}\wedge\bar{\eta}_{\mathcal{X}/S}}{\gamma^n_{\mathcal{X}/S}/n!}\cdot\pi^*\left(\frac{\mathrm{Vol}(\mathcal{X}/S)}{\|\eta_{\mathcal{X}/S}\|^2_{L^2}}\right)\right\}.\end{aligned}\tag{4.2}$$

Then we get

(4.3)

$$
\begin{aligned}
&-\frac{1}{12}\pi_*\left[c_1(\mathcal{X}/S)\,c_n(\mathcal{X}/S)\right]\\
&=-\frac{1}{12}\pi_*\left[-\pi^*\left\{\omega_{\mathrm{WP},\mathcal{X}/S}+dd^c\log\mathrm{Vol}(\mathcal{X}/S)\right\}\,c_n(\mathcal{X}/S)\right]\\
&\quad+\pi_*\left[-\frac{1}{12}dd^c\log\left\{\frac{(\sqrt{-1})^{n^2}\eta_{\mathcal{X}/S}\wedge\bar{\eta}_{\mathcal{X}/S}}{\gamma^n_{\mathcal{X}/S}/n!}\cdot\pi^*\left(\frac{\mathrm{Vol}(\mathcal{X}/S)}{\|\eta_{\mathcal{X}/S}\|^2_{L^2}}\right)\right\}\,c_n(\mathcal{X}/S)\right]\\
&=\frac{\chi(X)}{12}\,\omega_{\mathrm{WP},\,\mathcal{X}/S}+dd^c\log A(\mathcal{X}/S),
\end{aligned}
$$

where the first equality follows from (4.2), and the second one follows from the projection formula and the commutativity of dd^c and π_*.

Since the map $\pi\colon\mathcal{X}\to S$ is locally projective by Proposition 2.8, we may apply Theorem 3.5 to the family $\pi\colon\mathcal{X}\to S$. Then we deduce from (4.3) that

$$
\begin{aligned}
c_1(\overline{\lambda(\Omega^\bullet_{\mathcal{X}/S})})&=c_1(\lambda(\Omega^\bullet_{\mathcal{X}/S})_Q)-dd^c\log A(\mathcal{X}/S)\\
&=-\frac{1}{12}\pi_*[c_1(\mathcal{X}/S)\,c_n(\mathcal{X}/S)]-dd^c\log A(\mathcal{X}/S)\\
&=\frac{\chi(X)}{12}\,\omega_{\mathrm{WP},\,\mathcal{X}/S}.
\end{aligned}
$$

This completes the proof of Theorem 4.9. □

Theorem 4.10. *Let X be a smooth Calabi-Yau n-fold. The Hermitian metric $\|\cdot\|_{\lambda(\Omega^\bullet_X)}$ on $\lambda(\Omega^\bullet_X)$ is independent of the choice of a Kähler metric on X. In particular, the BCOV Hermitian line $\overline{\lambda(\Omega^\bullet_X)}$ is an invariant of X.*

Proof. Let $\sigma\in\lambda(\Omega^\bullet_X)\setminus\{0\}$. Let $\mathcal{X}=X\times\mathbb{P}^1\to\mathbb{P}^1$ be the trivial family over $\mathbb{P}^1$. Let γ_0, γ_∞ be arbitrary Kähler forms on X. Let $\gamma_{\mathcal{X}/\mathbb{P}^1}=\{\gamma_t\}_{t\in\mathbb{P}^1}$ be a C^∞-family of Kähler forms on X connecting γ_0 and γ_∞. Since $\omega_{\mathrm{WP},\mathcal{X}/\mathbb{P}^1}=0$, $\log\|\sigma\|^2_{\lambda(\Omega^\bullet_{\mathcal{X}/\mathbb{P}^1})}$ is a harmonic function on $\mathbb{P}^1$ by Theorem 4.9. Hence $\|\sigma\|_{\lambda(\Omega^\bullet_{\mathcal{X}/\mathbb{P}^1})}$ is a constant function on $\mathbb{P}^1$. This proves Theorem 4.10. □

4.4. **The BCOV invariant of Calabi-Yau threefolds**

In Subsection 4.4, we fix $n=3$. Hence X is a smooth Calabi-Yau threefold. Set $b_2(X):=\dim H^2(X,\mathbb{R})$. Let $c_X(\cdot,\cdot,\cdot)$ be the cubic form on $H^2(X,\mathbb{R})$ induced from the cup-product:

$$
c_X(\alpha,\beta,\gamma):=\frac{1}{(2\pi)^3}\int_X\alpha\wedge\beta\wedge\gamma,\qquad\alpha,\beta,\gamma\in H^2(X,\mathbb{R}).
$$

4.4.1. *The covolume of the cohomology lattice.* Let κ be a Kähler class on X. Let $\langle\cdot,\cdot\rangle_{L^2,\kappa}$ be the L^2-inner product on $H^2(X,\mathbb{R})$ with respect to κ, and let $\langle\cdot,\cdot\rangle_{L^2,\det\kappa}$ be the induced L^2-inner product on $\det H^2(X,\mathbb{R})$. Set $H^2(X,\mathbb{Z})_{\mathrm{fr}}:=H^2(X,\mathbb{Z})/\mathrm{Torsion}$.

Definition 4.11. For a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{b_2(X)}\}$ of $H^2(X,\mathbb{Z})_{\mathrm{fr}}$ over $\mathbb{Z}$, set

$$
\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),\kappa):=\det\left(\langle\mathbf{e}_i,\mathbf{e}_j\rangle_{L^2,\kappa}\right)=\langle\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{b_2(X)},\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{b_2(X)}\rangle_{L^2,\det\kappa}.
$$

Obviously, $\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),\kappa)$ is independent of the choice of a $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})_{\mathrm{fr}}$; it is the volume of the real torus $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})_{\mathrm{fr}}$ with respect to $\langle\cdot,\cdot\rangle_{L^2,\kappa}$. We can write $\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),\kappa)$ in terms of the cubic form c_X as follows:

Let L be the operator on $H^\bullet(X,\mathbb{R})$ defined by $L(\varphi)=\kappa\wedge\varphi$ for $\varphi\in H^\bullet(X,\mathbb{R})$.

Lemma 4.12. *The following identity holds*

$$\langle\alpha,\beta\rangle_{L^2,\kappa}=\frac{3}{2}\frac{c_X(\alpha,\kappa,\kappa)\,c_X(\beta,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}-c_X(\alpha,\beta,\kappa),\qquad \alpha,\beta\in H^2(X,\mathbb{R}).$$

In particular, $\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),\kappa)\in\mathbb{Q}$ *if* $\kappa\in H^2(X,\mathbb{Q})$.

Proof. Let $\varphi\in H^{1,1}(X,\mathbb{R})=H^2(X,\mathbb{R})$. By [57, Lemma 6.31], one has the orthogonal decomposition $H^{1,1}(X,\mathbb{R})=\ker(L^2)\oplus\mathbb{R}\kappa$ with respect to $\langle\cdot,\cdot\rangle_{L^2,\kappa}$. Since

$$\langle\varphi,\varphi\rangle_{L^2,\kappa}=\begin{cases}-c_X(\varphi,\varphi,\kappa) & (\varphi\in\ker(L^2))\\ \frac{1}{2}c_X(\varphi,\varphi,\kappa) & (\varphi\in\mathbb{R}\kappa)\end{cases}\tag{4.4}$$

by [57, Th. 6.32], we get the decomposition

$$\varphi=\left(\varphi-\frac{c_X(\varphi,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\right)+\frac{c_X(\varphi,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\in\ker(L^2)\oplus\mathbb{R}\kappa.\tag{4.5}$$

By (4.4), (4.5), we get

$$\begin{aligned}\langle\alpha,\beta\rangle_{L^2,\kappa}&=-c_X\left(\alpha-\frac{c_X(\alpha,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\,,\ \beta-\frac{c_X(\beta,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\,,\ \kappa\right)\\&\quad+\frac{1}{2}c_X\left(\frac{c_X(\alpha,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\,,\ \frac{c_X(\beta,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}\kappa\,,\ \kappa\right)\\&=\frac{3}{2}\frac{c_X(\alpha,\kappa,\kappa)\,c_X(\beta,\kappa,\kappa)}{c_X(\kappa,\kappa,\kappa)}-c_X(\alpha,\beta,\kappa).\end{aligned}$$

This proves the lemma. □

4.4.2. *The BCOV invariant.* Let us introduce the main object of this paper.

Definition 4.13. For a Kähler form γ on X, the *BCOV invariant* of (X,γ) is the real number defined by

$$\begin{aligned}\tau_{\mathrm{BCOV}}(X,\gamma):&=\mathrm{Vol}(X,\gamma)^{-3}\,\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma])^{-1}\,\mathcal{A}(X,\gamma)\,\mathcal{T}_{\mathrm{BCOV}}(X,\gamma)\\&=\mathrm{Vol}(X,\gamma)^{\frac{\chi(X)}{12}-3}\,\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma])^{-1}\\&\quad\times\exp\left[-\frac{1}{12}\int_X\log\left(\frac{\sqrt{-1}\,\eta\wedge\bar\eta}{\gamma^3/3!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|^2_{L^2}}\right)c_3(X,\gamma)\right]\mathcal{T}_{\mathrm{BCOV}}(X,\gamma).\end{aligned}$$

In the rest of Section 4, we derive a variational formula for the BCOV invariant.

4.4.3. *The curvature formula for the BCOV invariant.* Let $\pi\colon(\mathcal{X},X)\to(S,0)$ be a flat deformation of X which is induced from the Kuranishi family by a holomorphic map $\mu\colon(S,0)\to(\mathrm{Def}(X),[X])$. Let $\omega_{\mathrm{H},\mathcal{X}/S}$ be the $(1,1)$-form on S induced from the Hodge form on $\mathrm{Def}(X)$ via μ:

$$\omega_{\mathrm{H},\mathcal{X}/S}:=\mu^*\omega_{\mathrm{H}}.$$

Let $g_{\mathcal{X}/S}$ be a fiberwise-Kähler metric on $T\mathcal{X}/S$. Let γ_s be the Kähler form of $g_{\mathcal{X}/S}|_{X_s}$. Let $\tau_{\mathrm{BCOV}}(\mathcal{X}/S)$ be the function on S defined by

$$\tau_{\mathrm{BCOV}}(\mathcal{X}/S)(s):=\tau_{\mathrm{BCOV}}(X_s,\gamma_s),\qquad s\in S.$$

Theorem 4.14. *The following identity of* $(1,1)$*-forms on* $(S,0)$ *holds*

$$\begin{aligned} dd^c \log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) &= -\frac{\chi(X)}{12}\,\omega_{\mathrm{WP},\mathcal{X}/S} - \omega_{\mathrm{H},\mathcal{X}/S} \\ &= -\left(h^{1,2}(X) + \frac{\chi(X)}{12} + 3\right)\mu^*\omega_{\mathrm{WP}} - \mu^*\mathrm{Ric}\,\omega_{\mathrm{WP}}. \end{aligned}$$

Proof. We follow [60, Th. 5.6]. Let $\mathcal{A}(\mathcal{X}/S)$ and $\mathcal{T}_{\mathrm{BCOV}}(\mathcal{X}/S)$ be the C^∞ functions on S defined by

$$\mathcal{A}(\mathcal{X}/S)(s) := \mathcal{A}(X_s,\gamma_s), \qquad \mathcal{T}_{\mathrm{BCOV}}(\mathcal{X}/S)(s) := \mathcal{T}_{\mathrm{BCOV}}(X_s,\gamma_s)$$

for $s\in S$. By Theorems 3.5 and 4.9, we get

$$\begin{aligned} &-dd^c\log[\mathcal{A}(\mathcal{X}/S)\,\mathcal{T}_{\mathrm{BCOV}}(\mathcal{X}/S)] + \sum_{p,q\geq 0}(-1)^{p+q}p\,c_1(\det R^q\pi_*\Omega^p_{\mathcal{X}/S},\|\cdot\|_{L^2,g_{\mathcal{X}/S}}) \\ &= \frac{\chi(X)}{12}\mu^*\omega_{\mathrm{WP}}. \end{aligned}$$

Since $R^q\pi_*\Omega^p_{\mathcal{X}/S}\neq 0$ if and only if $p+q=3$ or $p=q$, we deduce from Proposition 4.6 that

$$\begin{aligned} &-dd^c\log[\mathcal{A}(\mathcal{X}/S)\,\mathcal{T}_{\mathrm{BCOV}}(\mathcal{X}/S)] + \sum_{p>0}p\,c_1(\det R^p\pi_*\Omega^p_{\mathcal{X}/S},\|\cdot\|_{L^2,g_{\mathcal{X}/S}}) \\ &-\big(\mu^*\mathrm{Ric}\,\omega_{\mathrm{WP}} + h^{1,2}(X)\,\mu^*\omega_{\mathrm{WP}}\big) - 3\mu^*\omega_{\mathrm{WP}} \\ &= \frac{\chi(X)}{12}\mu^*\omega_{\mathrm{WP}}. \end{aligned} \tag{4.6}$$

Define a function $\mathrm{Vol}_{L^2}(H^2(\mathcal{X}/S,\mathbb{Z}))$ on S by

$$\mathrm{Vol}_{L^2}(H^2(\mathcal{X}/S,\mathbb{Z}))(s) := \mathrm{Vol}_{L^2}(H^2(X_s,\mathbb{Z}),[\gamma_s]), \qquad s\in S.$$

Since $\pi\colon\mathcal{X}\to S$ is induced from the Kuranishi family, there exist holomorphic line bundles $\mathcal{L}_1,\ldots,\mathcal{L}_{b_2(X)}$ on $\mathcal{X}$ by Lemma 2.16 such that $c_1(\mathcal{L}_i)|_X = \mathbf{e}_i$ for $1\leq i\leq b_2(X)$, and such that $\mathfrak{C}_1(\mathcal{L}_1)\wedge\cdots\wedge\mathfrak{C}_1(\mathcal{L}_{b_2(X)})$ is a nowhere vanishing holomorphic section of $R^1\pi_*\Omega^1_{\mathcal{X}/S}$. Then

$$\|\mathfrak{C}_1(\mathcal{L}_1)\wedge\cdots\wedge\mathfrak{C}_1(\mathcal{L}_{b_2(X)})\|^2_{L^2,g_{\mathcal{X}/S}} = \mathrm{Vol}_{L^2}(H^2(\mathcal{X}/S,\mathbb{Z})). \tag{4.7}$$

By the Serre duality and (3.5), $\mathbf{1}_{(1,1),(2,2)}\otimes(\mathfrak{C}_1(\mathcal{L}_1)\wedge\cdots\wedge\mathfrak{C}_1(\mathcal{L}_{b_2(X)}))^{-1}$ is a nowhere vanishing holomorphic section of $R^2\pi_*\Omega^2_{\mathcal{X}/S}$ such that

$$\|\mathbf{1}_{(1,1),(2,2)}\otimes(\mathfrak{C}_1(\mathcal{L}_1)\wedge\cdots\wedge\mathfrak{C}_1(\mathcal{L}_{b_2(X)}))^{-1}\|^2_{L^2,g_{\mathcal{X}/S}} = \mathrm{Vol}_{L^2}(H^2(\mathcal{X}/S,\mathbb{Z}))^{-1}. \tag{4.8}$$

Let $\mathrm{Vol}(\mathcal{X}/S,\gamma_{\mathcal{X}/S})$ be the function on S defined by

$$\mathrm{Vol}(\mathcal{X}/S)(s) := \mathrm{Vol}(X_s,\gamma_s).$$

Then $\frac{\gamma^3_{\mathcal{X}/S}}{3!\mathrm{Vol}(\mathcal{X}/S)}$ is a nowhere vanishing holomorphic section of $R^3\pi_*\Omega^3_{\mathcal{X}/S}$ such that

$$\left\|\frac{\gamma^3_{\mathcal{X}/S}}{3!\mathrm{Vol}(\mathcal{X}/S)}\right\|^2_{L^2,g_{\mathcal{X}/S}} = \mathrm{Vol}(\mathcal{X}/S)^{-1}. \tag{4.9}$$

Substituting (4.7), (4.8), (4.9) into (4.6), we get the equation:
(4.10)

$$-dd^c \log[\mathcal{A}(\mathcal{X}/S)\,\mathcal{T}_{\mathrm{BCOV}}(\mathcal{X}/S)] + dd^c \log \mathrm{Vol}_{L^2}(H^2(\mathcal{X}/S,\mathbb{Z})) + 3dd^c \log \mathrm{Vol}(\mathcal{X}/S)$$
$$= \left(h^{1,2}(X) + \frac{\chi(X)}{12} + 3\right)\mu^*\omega_{\mathrm{WP}} + \mu^*\mathrm{Ric}\,\omega_{\mathrm{WP}}.$$

The theorem follows from the definition of the BCOV invariant and (4.10). □

Remark 4.15. If we follow the mirror symmetry and if $X^\vee$ is the mirror Calabi-Yau threefold of X, the coefficient of $\mu^*\omega_{\mathrm{WP}}$ in (4.10) is compatible with that of [6, Eq. (14)] since $h^{1,1}(X^\vee) = h^{1,2}(X)$ and $\chi(X^\vee) = -\chi(X)$.

For a higher dimensional analogue of Theorem 4.14, we refer to [17].

Theorem 4.16. *The BCOV invariant $\tau_{\mathrm{BCOV}}(X,\gamma)$ is independent of the choice of a Kähler metric on X. In particular, $\tau_{\mathrm{BCOV}}(X,\gamma)$ is an invariant of X.*

Proof. Let $\mathcal{X} = X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the trivial family over $\mathbb{P}^1$. Let γ_0, γ_∞ be arbitrary Kähler forms on X. Let $\gamma_{\mathcal{X}/\mathbb{P}^1} = \{\gamma_t\}_{t\in\mathbb{P}^1}$ be a C^∞-family of Kähler forms on X connecting γ_0 and γ_∞. Since $\mu^*\omega_{\mathrm{WP}}$ and $\mu^*\mathrm{Ric}(\omega_{\mathrm{WP}})$ are independent of t, $\log\tau_{\mathrm{BCOV}}(\mathcal{X}/\mathbb{P}^1)$ is a harmonic function on $\mathbb{P}^1$ by Theorem 4.14. Hence $\tau_{\mathrm{BCOV}}(\mathcal{X}/\mathbb{P}^1)$ is a constant function on $\mathbb{P}^1$. □

After Theorem 4.16, we shall write $\tau_{\mathrm{BCOV}}(X)$ for $\tau_{\mathrm{BCOV}}(X,\gamma)$ in the rest of this paper.

5. The singularity of the Quillen metric on the BCOV bundle

In Section 5, we fix the following notation: Let $\mathcal{X}$ be a compact Kähler manifold of dimension $n+1$ and let S be a compact Riemann surface. Let $\pi\colon \mathcal{X} \to S$ be a surjective holomorphic map, and we do *not* assume that a general fiber of π is Calabi-Yau.

Let Σ_π be the critical locus of π, and set

$$\mathcal{D} := \pi(\Sigma_\pi), \qquad S^o := S \setminus \mathcal{D}, \qquad \mathcal{X}^o := \pi^{-1}(S^o), \qquad \pi^o := \pi|_{\mathcal{X}^o}.$$

Then $\pi^o\colon \mathcal{X}^o \to S^o$ is a holomorphic family of compact complex manifolds, and $\Omega^1_{\mathcal{X}^o/S^o}$ is a holomorphic vector bundle of rank n over $\mathcal{X}^o$.

As in Sections 3 and 4, we have the holomorphic line bundles on S^o:

$$\lambda(\Omega^p_{\mathcal{X}^o/S^o}) = \otimes_{q=0}^n (\det R^q\pi_*\Omega^p_{\mathcal{X}^o/S^o})^{(-1)^q}, \qquad \lambda(\Omega^\bullet_{\mathcal{X}^o/S^o}) = \otimes_{p=0}^n \lambda(\Omega^p_{\mathcal{X}^o/S^o})^{(-1)^p p}.$$

In this section, we construct holomorphic extensions of $\lambda(\Omega^p_{\mathcal{X}^o/S^o})$ and $\lambda(\Omega^\bullet_{\mathcal{X}^o/S^o})$ from S^o to S, and we study the singularity of the corresponding Quillen metrics.

5.1. The Kähler extension of the determinant line bundles

Since $\Omega^1_{\mathcal{X}/S} = \Omega^1_{\mathcal{X}}/\pi^*\Omega^1_S$, we have the following complex of coherent sheaves on $\mathcal{X}$, which is acyclic on $\mathcal{X}$ (cf. [33, p.94 l.12-l.16]):

$$0 \longrightarrow \pi^*\Omega^1_S \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/S} \longrightarrow 0.$$

Definition 5.1. (i) For $p > 0$, let $\mathcal{E}^p_{\mathcal{X}/S}$ be the complex of holomorphic vector bundles on $\mathcal{X}$ defined by

$$\mathcal{E}^p_{\mathcal{X}/S}\colon (\pi^*\Omega^1_S)^{\otimes p} \longrightarrow \Omega^1_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-1)} \longrightarrow \cdots \longrightarrow \Omega^{p-1}_{\mathcal{X}} \otimes \pi^*\Omega^1_S \longrightarrow \Omega^p_{\mathcal{X}},$$

where the maps $\Omega^i_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-i)} \to \Omega^{i+1}_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-i-1)}$ are given by

$$\omega \otimes (\pi^*\xi)^{\otimes(p-i)} \mapsto (\omega \wedge \pi^*\xi) \otimes (\pi^*\xi)^{\otimes(p-i-1)}, \qquad \omega \in \Omega^i_{\mathcal{X}}, \quad \xi \in \Omega^1_S.$$

For $p = 0$, set $\mathcal{E}^0_{\mathcal{X}/S} \colon 0 \to \mathcal{O}_{\mathcal{X}} \to 0$.

(ii) For $p \geq 0$, let $\mathcal{F}^p_{\mathcal{X}/S}$ be the complex of coherent sheaves on $\mathcal{X}$ defined by

$$\mathcal{F}^p_{\mathcal{X}/S} \colon 0 \longrightarrow \mathcal{E}^p_{\mathcal{X}/S} \xrightarrow{r} \Omega^p_{\mathcal{X}/S} \longrightarrow 0,$$

where $r \colon \Omega^p_{\mathcal{X}} \to \Omega^p_{\mathcal{X}/S}$ is the quotient map for $p > 0$ and the identity map for $p = 0$.

Since $\mathrm{rk}(\pi^*\Omega^1_S) = 1$, $\mathcal{F}^p_{\mathcal{X}/S}$ is acyclic on $\mathcal{X} \setminus \Sigma_\pi$ for $p > 1$ and on $\mathcal{X}$ for $p = 0, 1$.

Definition 5.2. (i) Let $\lambda(\mathcal{E}^p_{\mathcal{X}/S})$ be the holomorphic line bundle on S defined by

$$\lambda(\mathcal{E}^p_{\mathcal{X}/S}) := \bigotimes_{i=0}^{p} \lambda(\Omega^{p-i}_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes i})^{(-1)^i}.$$

(ii) Let $\lambda(\Omega^\bullet_{\mathcal{X}/S})$ be the holomorphic line bundle on S defined by

$$\lambda(\Omega^\bullet_{\mathcal{X}/S}) := \bigotimes_{p \geq 0} \lambda(\mathcal{E}^p_{\mathcal{X}/S})^{(-1)^p p}.$$

We call $\lambda(\mathcal{E}^p_{\mathcal{X}/S})$ and $\lambda(\Omega^\bullet_{\mathcal{X}/S})$ the *Kähler extensions* of $\lambda(\Omega^p_{\mathcal{X}^o/S^o})$ and $\lambda(\Omega^\bullet_{\mathcal{X}^o/S^o})$ from S^o to S, respectively.

Since $\mathcal{F}^p_{\mathcal{X}/S}$ is acyclic on $\mathcal{X} \setminus \Sigma_\pi$, we have the canonical isomorphisms of holomorphic line bundles on S^o:

$$\lambda(\Omega^p_{\mathcal{X}^o/S^o}) \cong \lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{S^o}, \qquad \lambda(\Omega^\bullet_{\mathcal{X}^o/S^o}) \cong \lambda(\Omega^\bullet_{\mathcal{X}/S})|_{S^o}.$$

Let $g_{\mathcal{X}}$ be a Kähler metric on $\mathcal{X}$. Let $g_{\mathcal{X}/S} := g_{\mathcal{X}}|_{T\mathcal{X}/S}$ be the Hermitian metric on $T\mathcal{X}/S|_{\mathcal{X}\setminus\Sigma_\pi}$ induced from $g_{\mathcal{X}}$. Then $g_{\mathcal{X}/S}$ (resp. $g_{\mathcal{X}}$) induces the Hermitian metric $g_{\Omega^p_{\mathcal{X}/S}}$ (resp. $g_{\Omega^p_{\mathcal{X}}}$) on $\Omega^p_{\mathcal{X}/S}|_{\mathcal{X}\setminus\Sigma_\pi}$ (resp. $\Omega^p_{\mathcal{X}}$) for all $p \geq 0$.

Following Bismut [9] and Yoshikawa [61], we determine the singularity of the Quillen metric on $\lambda(\Omega^p_{\mathcal{X}^o/S^o})$ near $\mathcal{D}$ with respect to the Kähler extension and with respect to the metrics $g_{\mathcal{X}/S}$, $g_{\Omega^p_{\mathcal{X}/S}}$.

5.2. Three Quillen metrics on the extended BCOV bundles

Let $0 \in \mathcal{D}$. Let $(\mathcal{U}, t)$ be a coordinate neighborhood of 0 in S centered at 0 such that $\mathcal{U} \cong \Delta$ and $\mathcal{U} \cap \mathcal{D} = \{0\}$. We set $\mathcal{U}^o := \mathcal{U} \setminus \mathcal{D} = \mathcal{U} \setminus \{0\}$.

Let k_S be a Hermitian metric on Ω^1_S such that $k_S(dt, dt) = 1$ on $\mathcal{U}$. Then π^*k_S is a Hermitian metric on $\pi^*\Omega^1_S$. Let $g_{\pi^*\Omega^1_S}$ be the Hermitian metric on $\pi^*\Omega^1_S|_{\mathcal{X}\setminus\Sigma_\pi}$ induced from $g_{\Omega^1_{\mathcal{X}}}$ by the inclusion $\pi^*\Omega^1_S \subset \Omega^1_{\mathcal{X}}$. Since

$$\pi^*k_S(d\pi, d\pi) = \pi^*\{k_S(dt, dt)\} = 1, \qquad g_{\pi^*\Omega^1_S}(d\pi, d\pi) = g_{\Omega^1_{\mathcal{X}}}(d\pi, d\pi) = \|d\pi\|^2$$

on $\pi^{-1}(\mathcal{U})$, the following identity holds on $\pi^{-1}(\mathcal{U})$:

$$g_{\pi^*\Omega^1_S} = \|d\pi\|^2\, \pi^*k_S.$$

We define three Quillen metrics on the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}^o}$ as follows.

Definition 5.3. (i) Let $\|\cdot\|^2_{\lambda(\Omega^p_{\mathcal{X}^o/S^o}),Q,g_{\mathcal{X}/S}}$ be the Quillen metric on $\lambda(\Omega^p_{\mathcal{X}^o/S^o})|_{\mathcal{U}^o}$ with respect to $g_{\mathcal{X}/S}$ and $g_{\Omega^p_{\mathcal{X}/S}}$. Let $\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}$ be the Quillen metric on $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}^o}$ induced from $\|\cdot\|^2_{\lambda(\Omega^p_{\mathcal{X}^o/S^o}),Q,g_{\mathcal{X}/S}}$ by the canonical isomorphism $\lambda(\Omega^p_{\mathcal{X}^o/S^o})|_{\mathcal{U}^o} \cong \lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}^o}$:

$$\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}} := \|\cdot\|^2_{\lambda(\Omega^p_{\mathcal{X}^o/S^o}),Q,g_{\mathcal{X}/S}}.$$

(ii) Let $\|\cdot\|^2_{\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i}),Q,\pi^*k_S}$ be the Quillen metric on $\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i})|_{\mathcal{U}^o}$ with respect to $g_{\mathcal{X}/S}$ and $g_{\Omega^{p-i}_{\mathcal{X}}}\otimes\pi^*k_S$. Set

$$\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S} := \bigotimes_{i=0}^{p}\|\cdot\|^{2(-1)^i}_{\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i}),Q,\pi^*k_S}.$$

(iii) Let $\|\cdot\|^2_{\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i}),Q,h_{\pi^*\Omega^1_S}}$ be the Quillen metric on $\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i})|_{\mathcal{U}^o}$ with respect to $g_{\mathcal{X}/S}$ and $g_{\Omega^{p-i}_{\mathcal{X}}}\otimes g_{\pi^*\Omega^1_S}$. Set

$$\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}} := \bigotimes_{i=0}^{p}\|\cdot\|^{2(-1)^i}_{\lambda(\Omega^{p-i}_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes i}),Q,g_{\pi^*\Omega^1_S}}.$$

When $p=0$, we have the following relations

$$\|\cdot\|^2_{\lambda(\mathcal{E}^0_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}} = \|\cdot\|^2_{\lambda(\mathcal{E}^0_{\mathcal{X}/S}),Q,\pi^*k_S} = \|\cdot\|^2_{\lambda(\mathcal{E}^0_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}} = \|\cdot\|^2_{\lambda(\mathcal{O}_{\mathcal{X}}),Q,g_{\mathcal{X}/S}}.$$

We shall prove that $\log\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}$ has logarithmic singularities at $0\in\mathcal{D}$, whose coefficients are determined by the resolution data of the Gauss map.

5.3. The Gauss maps and their resolutions

Let $\Pi\colon \mathbb{P}(\Omega^1_{\mathcal{X}}) \to \mathcal{X}$ be the projective bundle associated with the holomorphic cotangent bundle $\Omega^1_{\mathcal{X}}$. Let $\Pi^\vee\colon \mathbb{P}(T\mathcal{X}) \to \mathcal{X}$ be the projective bundle associated with the holomorphic tangent bundle $T\mathcal{X}$. Then the fiber $\mathbb{P}(T_x\mathcal{X})^\vee$ is the set of all hyperplanes of $T_x\mathcal{X}$ containing $0_x \in T_x\mathcal{X}$. We have $\mathbb{P}(\Omega^1_{\mathcal{X}}) \cong \mathbb{P}(T\mathcal{X})^\vee$.

We define the Gauss maps $\nu\colon \mathcal{X}\setminus\Sigma_\pi \to \mathbb{P}(\Omega^1_{\mathcal{X}})$ and $\mu\colon \mathcal{X}\setminus\Sigma_\pi \to \mathbb{P}(T\mathcal{X})^\vee$ by

$$\nu(x) := [d\pi_x] = \left[\sum_{i=0}^{n}\frac{\partial\pi}{\partial z_i}(x)\,dz_i\right], \qquad \mu(x) := [T_xX_{\pi(x)}].$$

Then $\nu=\mu$ under the canonical isomorphism $\mathbb{P}(\Omega^1_{\mathcal{X}}) \cong \mathbb{P}(T\mathcal{X})^\vee$.

Let $L := \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathcal{X}})}(-1) \subset \Pi^*\Omega^1_{\mathcal{X}}$ be the tautological line bundle over $\mathbb{P}(\Omega^1_{\mathcal{X}})$, and set $Q := \Pi^*\Omega^1_{\mathcal{X}}/L$. Then we have the following exact sequences $\mathcal{S}$ of holomorphic vector bundles on $\mathbb{P}(\Omega^1_{\mathcal{X}})$:

$$\mathcal{S}\colon 0 \longrightarrow L \longrightarrow \Pi^*\Omega^1_{\mathcal{X}} \longrightarrow Q \longrightarrow 0.$$

Let $p\le n$. Since $\mathrm{rk}(L)=1$, this induces the following exact sequence of holomorphic vector bundles on $\mathbb{P}(\Omega^1_{\mathcal{X}})$:

$$\mathcal{K}^p\colon 0 \longrightarrow L^p \longrightarrow \Pi^*\Omega^1_{\mathcal{X}}\otimes L^{p-1} \longrightarrow \cdots \longrightarrow \Pi^*\Omega^{p-1}_{\mathcal{X}}\otimes L \longrightarrow \Pi^*\Omega^p_{\mathcal{X}} \longrightarrow \bigwedge^p Q \longrightarrow 0,$$

where $\Pi^*\Omega^p_{\mathcal{X}} \to \bigwedge^p Q$ is the quotient map and $\Pi^*\Omega^i_{\mathcal{X}} \otimes L^{p-i} \to \Pi^*\Omega^{i+1}_{\mathcal{X}} \otimes L^{p-i-1}$ is given by $\omega \otimes \sigma^{\otimes(p-i)} \mapsto (\omega \wedge \sigma) \otimes \sigma^{\otimes(p-i-1)}$ for $\omega \in \Pi^*\Omega^1_{\mathcal{X}}$ and $\sigma \in L$. Then

$$\mathcal{F}^p_{\mathcal{X}/S} = \nu^*\mathcal{K}^p.$$

Similarly, let $H := \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathcal{X}})}(1)$, and let U be the universal hyperplane bundle of $(\Pi^\vee)^*T\mathcal{X}$. Then the dual of $\mathcal{S}$ is given by

$$\mathcal{S}^\vee \colon 0 \longrightarrow U \longrightarrow (\Pi^\vee)^*T\mathcal{X} \longrightarrow H \longrightarrow 0.$$

Since $T_x\mathcal{X}/S = \{v \in T_x\mathcal{X};\, d\pi_x(v) = 0\}$, we have

$$T\mathcal{X}/S = \mu^*U.$$

Let g_U be the Hermitian metric on U induced from $(\Pi^\vee)^*g_{\mathcal{X}}$, and let g_H be the Hermitian metric on H induced from $(\Pi^\vee)^*g_{\mathcal{X}}$ by the C^∞-isomorphism $H \cong U^\perp$.

Let g_L be the Hermitian metric on L induced from $\Pi^*g_{\Omega^1_{\mathcal{X}}}$ by the inclusion $L \subset \Pi^*\Omega^1_{\mathcal{X}}$. Let g_Q be the Hermitian metric on Q induced from $\Pi^*g_{\Omega^1_{\mathcal{X}}}$ by the C^∞-isomorphism $Q \cong L^\perp$. We consider the Hermitian metric $g_{\Pi^*\Omega^i_{\mathcal{X}}\otimes L^{p-i}}$ on $\Pi^*\Omega^i_{\mathcal{X}} \otimes L^{p-i}$ induced from $\Pi^*g_{\Omega^1_{\mathcal{X}}}$, g_L, and we consider the Hermitian metric $g_{\wedge^p Q}$ on $\bigwedge^p Q$ induced from g_Q. We define $\overline{\mathcal{K}}^p$ to be the exact sequence $\mathcal{K}^p$ equipped with the Hermitian metrics $\{g_{\Pi^*\Omega^i_{\mathcal{X}}\otimes L^{p-i}}\}$ and $g_{\wedge^p Q}$. Then we have the following isomorphisms of Hermitian vector bundles over $\mathcal{X} \setminus \Sigma_\pi$:

$$\overline{\mathcal{F}}^p_{\mathcal{X}/S} = \nu^*\overline{\mathcal{K}}^p, \qquad (T\mathcal{X}/S, g_{\mathcal{X}/S}) = \mu^*(U, g_U). \tag{5.1}$$

Since $d\pi$ is a nowhere vanishing holomorphic section of $\nu^*L|_{\mathcal{X}\setminus\Sigma_\pi}$, we get the following equation on $\mathcal{X} \setminus \Sigma_\pi$

$$-dd^c \log \|d\pi\|^2 = \nu^*c_1(L, g_L).$$

Since Σ_π is a proper analytic subset of $\mathcal{X}$, the Gauss maps $\nu\colon \mathcal{X} \setminus \Sigma_\pi \to \mathbb{P}(\Omega^1_{\mathcal{X}})$ and $\mu\colon \mathcal{X} \setminus \Sigma_\pi \to \mathbb{P}(T\mathcal{X})^\vee$ extend to meromorphic maps $\nu\colon \mathcal{X} \dashrightarrow \mathbb{P}(\Omega^1_{\mathcal{X}})$ and $\mu\colon \mathcal{X} \dashrightarrow \mathbb{P}(T\mathcal{X})^\vee$ by e.g. [43, Th. 4.5.3]. By Hironaka, there exist a projective algebraic manifold $\widetilde{\mathcal{X}}$, a divisor of normal crossing $E \subset \mathcal{X}$, a birational holomorphic map $q\colon \widetilde{\mathcal{X}} \to \mathcal{X}$, and holomorphic maps $\widetilde{\nu}\colon \widetilde{\mathcal{X}} \to \mathbb{P}(\Omega^1_{\mathcal{X}})$ and $\widetilde{\mu}\colon \widetilde{\mathcal{X}} \to \mathbb{P}(T\mathcal{X})^\vee$ satisfying the following conditions:

(i) $q|_{\widetilde{\mathcal{X}}\setminus q^{-1}(\Sigma_\pi)}\colon \widetilde{\mathcal{X}} \setminus q^{-1}(\Sigma_\pi) \to \mathcal{X} \setminus \Sigma_\pi$ is an isomorphism;
(ii) $q^{-1}(\Sigma_\pi) = E$;
(iii) $\widetilde{\nu} = \nu \circ q$ and $\widetilde{\mu} = \mu \circ q$ on $\widetilde{\mathcal{X}} \setminus E$.

By (iii), we have $\widetilde{\nu} = \widetilde{\mu}$ under the canonical isomorphism $\mathbb{P}^1(\Omega^1_{\mathcal{X}}) = \mathbb{P}(T\mathcal{X})^\vee$.

We set $\widetilde{\pi} := \pi \circ q$ and $\widetilde{X}_t := \widetilde{\pi}^{-1}(t)$ for $t \in S$. Similarly, we set $E_b := E \cap \widetilde{X}_b$ for $b \in \mathcal{D}$. Then $E = \amalg_{b\in\mathcal{D}} E_b$, because $E = q^{-1}(\Sigma_\pi) \subset \widetilde{\pi}^{-1}(\mathcal{D})$.

5.4. **The singularity of Quillen metrics**

After Barlet [3], we define a subspace of $C^0(\mathcal{U})$ by

$$\mathcal{B}(\mathcal{U}) := C^\infty(\mathcal{U}) \oplus \bigoplus_{r\in\mathbb{Q}\cap(0,1]} \bigoplus_{k=0}^{n} |t|^{2r}(\log|t|)^k \cdot C^\infty(\mathcal{U}).$$

A function $\varphi(t) \in \mathcal{B}(\mathcal{U})$ has an asymptotic expansion at $0 \in \mathcal{D}$, i.e., there exist $r_1, \dots, r_m \in \mathbb{Q} \cap (0,1]$ and $f_0, f_{l,k} \in C^\infty(\mathcal{U})$, $l = 1, \dots, m$, $k = 0, \dots, n$, such

that $\varphi(t) = f_0(t) + \sum_{l=1}^{m} \sum_{k=0}^{n} |t|^{2r_l} (\log |t|)^k f_{l,k}(t)$ as $t \to 0$. In what follows, if $f(t), g(t) \in C^\infty(\mathcal{U}^o)$ satisfies $f(t) - g(t) \in \mathcal{B}(\mathcal{U})$, we write

$$f \equiv_{\mathcal{B}} g.$$

The purpose of Section 5 is to prove the following:

Theorem 5.4. *Let σ_p be a nowhere vanishing C^∞ section of the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}$. Then*

$$\log \|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}} \equiv_{\mathcal{B}}$$
$$\left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.$$

The proof of Theorem 5.4 is divided into the following three intermediary results, whose proofs shall be given in the subsections below:

Proposition 5.5. *The following identity of functions on $\mathcal{U}$ holds*

$$\log(\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}} / \| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}) \equiv_{\mathcal{B}} 0.$$

Proposition 5.6. *The following identity of functions on $\mathcal{U}$ holds*

$$\log \left(\frac{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}}{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S}} \right) \equiv_{\mathcal{B}}$$
$$\left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{1 - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.$$

Proposition 5.7. *The following identity of functions on $\mathcal{U}$ holds*

$$\log \|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S} \equiv_{\mathcal{B}}$$
$$\left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.$$

Proof of Theorem 5.4. By Propositions 5.5, 5.6, and 5.7, we get

$$\begin{aligned}
&\log \|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}} = \\
&\log \left(\frac{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}}{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}} \right) + \log \left(\frac{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}}{\| \cdot \|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S}} \right) + \log \|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S} \\
&\equiv_{\mathcal{B}} \left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{1 - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2 \\
&\quad + \left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2 \\
&\equiv_{\mathcal{B}} \left(\int_{E_0} \sum_{j=0}^{p} (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.
\end{aligned}$$

This proves the theorem. $\square$

5.5. **Proof of Proposition 5.5**

Let $g_{\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}}$ be the Hermitian metric on $\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}$ induced from $g_{\mathcal{X}}$, $g_{\pi^*\Omega^1_S}$. We define $\overline{\mathcal{F}}^p_{\mathcal{X}/S}$ to be the complex of holomorphic vector bundles $\mathcal{F}^p_{\mathcal{X}/S}$ equipped with the Hermitian metrics $g_{\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}}$ on $\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}$ and $g_{\Omega^p_{\mathcal{X}/S}}$ on $\Omega^p_{\mathcal{X}/S}$.

Let π_* (resp. $\widetilde{\pi}_*$) be the integration along the fibers of π (resp. $\widetilde{\pi}$). For a C^∞ differential form ψ on $\widetilde{\mathcal{X}}$, one has $\widetilde{\pi}_*(\psi)^{(0,0)}\in\mathcal{B}(\mathcal{U})$ by [3, Th. 4bis].

Since $\mathcal{F}^p_{\mathcal{X}/S}$ is acyclic on $\mathcal{X}^o$, the following identity of C^∞ functions on S^o holds by the anomaly formula [11, Th. 0.3]:

$$\log\left(\frac{\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}}{\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}}\right)=\pi_*\left(\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}^p_{\mathcal{X}/S})\right)^{(0,0)}. \tag{5.2}$$

By (5.1), the following identity of C^∞ differential forms on $\mathcal{X}\setminus\Sigma_\pi$ holds:

$$\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}^p_{\mathcal{X}/S})|_{\mathcal{X}\setminus\Sigma_\pi}=\mu^*\mathrm{Td}(U,g_U)\,\nu^*\widetilde{\mathrm{ch}}(\overline{\mathcal{K}}^p).$$

Since $q_*=(q^{-1})^*$ on $\widetilde{\mathcal{X}}\setminus q^{-1}(\Sigma_\pi)$, this yields the following identity on $\mathcal{X}\setminus\Sigma_\pi$:

$$\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}^p_{\mathcal{X}/S})|_{\mathcal{X}\setminus\Sigma_\pi}=(q)_*\left\{\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\widetilde{\mathrm{ch}}(\overline{\mathcal{K}}^p)\right\}.$$

Hence we get the following equation of C^∞ functions on S^o:

$$\pi_*\left(\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}^p_{\mathcal{X}})|_{\mathcal{X}\setminus\Sigma_\pi}\right)^{(0,0)}=\left[\widetilde{\pi}_*\left\{\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\widetilde{\mathrm{ch}}(\overline{\mathcal{K}}^p)\right\}\right]^{(0,0)}. \tag{5.3}$$

Since $\{\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\widetilde{\mathrm{ch}}(\overline{\mathcal{K}}^p)\}^{(n,n)}$ is a C^∞ (n,n)-form on $\widetilde{\mathcal{X}}$ and since the projection $\widetilde{\pi}\colon\widetilde{\mathcal{X}}\to S$ is proper and holomorphic, the right hand side of (5.3) lies in $\mathcal{B}(\mathcal{U})$ by [3, Th. 4bis], which, together with (5.2), (5.3), yields the result. $\square$

5.6. **Proof of Proposition 5.6**

For $0\le i\le p$, we deduce from the anomaly formula [11, Th. 0.3] that

$$\begin{aligned}
&\log\left(\frac{\|\cdot\|^2_{\lambda(\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}),Q,g_{\pi^*\Omega^1_S}}}{\|\cdot\|^2_{\lambda(\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}),Q,\pi^*k_S}}\right)\\
&=\pi_*\left(\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\mathrm{ch}(\Omega^i_{\mathcal{X}},g_{\mathcal{X}})\,\widetilde{\mathrm{ch}}((\pi^*\Omega^1_S)^{\otimes(p-i)};\,\pi^*k_S,\,g_{\pi^*\Omega^1_S})\right)^{(0,0)}\\
&=\pi_*\left(\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\mathrm{ch}(\Omega^i_{\mathcal{X}},g_{\mathcal{X}})\,\widetilde{\mathrm{ch}}((\pi^*\Omega^1_S)^{\otimes(p-i)};\,\pi^*k_S,\,\|d\pi\|^2\pi^*k_S)\right)^{(0,0)}.
\end{aligned} \tag{5.4}$$

Since $\nu^* c_1(L, g_L)|_{\mathcal{X}\setminus\Sigma_\pi} = -dd^c \log\|d\pi\|^2$ and $c_1(\Omega^1_S, k_S) = 0$ on $\mathcal{U}$, we deduce from (3.7) that

(5.5)
$$
\begin{aligned}
&\widetilde{\mathrm{ch}}((\pi^*\Omega^1_S)^{\otimes l};\, \pi^* k_S^{\otimes l},\, \|d\pi\|^{2l}\pi^* k_S^{\otimes l})\Big|_{\pi^{-1}(\mathcal{U})\setminus\Sigma_\pi} \\
&= \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{a+b=m-1} c_1\left((\pi^*\Omega^1_S)^{\otimes l}, \pi^* k_S^{\otimes l}\right)^a\, c_1\left((\pi^*\Omega^1_S)^{\otimes l}, \|d\pi\|^{2l}\pi^* k_S^{\otimes l}\right)^b \log\|d\pi\|^{2l} \\
&= \sum_{m=1}^{\infty} \frac{1}{m!}(-dd^c \log\|d\pi\|^{2l})^{m-1} \log\|d\pi\|^{2l} = \frac{e^{l\nu^* c_1(L,g_L)} - 1}{\nu^* c_1(L, g_L)} \log\|d\pi\|^2.
\end{aligned}
$$

By substituting (5.5) and $\mathrm{Td}(T\mathcal{X}/S, g_{\mathcal{X}/S}) = \mu^*\mathrm{Td}(U, g_U)$ into (5.4), we get

(5.6)
$$
\begin{aligned}
&\log\left(\frac{\|\cdot\|^2_{\lambda(\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}),Q,g_{\pi^*\Omega^1_S}}}{\|\cdot\|^2_{\lambda(\Omega^i_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-i)}),Q,\pi^* k_S}}\right)\Bigg|_{\mathcal{U}^o} \\
&= \pi_*\left\{\mu^*\mathrm{Td}(U, g_U)\,\mathrm{ch}(\Omega^i_{\mathcal{X}}, g_{\mathcal{X}})\, \frac{e^{(p-i)\nu^* c_1(L,g_L)} - 1}{\nu^* c_1(L, g_L)} \log\|d\pi\|^2\right\}^{(0,0)} \\
&= \widetilde{\pi}_*\left\{\widetilde{\mu}^*\mathrm{Td}(U, g_U)\, q^*\mathrm{ch}(\Omega^i_{\mathcal{X}}, g_{\mathcal{X}})\, \frac{e^{(p-i)\widetilde{\nu}^* c_1(L,g_L)} - 1}{\widetilde{\nu}^* c_1(L, g_L)}\, q^*(\log\|d\pi\|^2)\right\}^{(0,0)},
\end{aligned}
$$

which yields that

(5.7)
$$
\begin{aligned}
&\log\left(\frac{\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\pi^*\Omega^1_S}}}{\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^* k_S}}\right)\Bigg|_{\mathcal{U}^o} = \\
&\sum_{j=0}^{p}(-1)^{p-j} \log\left(\frac{\|\cdot\|^2_{\lambda(\Omega^j_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-j)}),Q,g_{\pi^*\Omega^1_S}}}{\|\cdot\|^2_{\lambda(\Omega^j_{\mathcal{X}}\otimes(\pi^*\Omega^1_S)^{\otimes(p-j)}),Q,\pi^* k_S}}\right) = \\
&\widetilde{\pi}_*\left[q^*(\log\|d\pi\|^2) \sum_{j=0}^{p}(-1)^{p-j}\widetilde{\mu}^*\mathrm{Td}(U, g_U)\,\widetilde{\nu}^*\left\{\frac{e^{(p-j)c_1(L,g_L)} - 1}{c_1(L, g_L)}\right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}, g_{\mathcal{X}})\right]^{(0,0)}.
\end{aligned}
$$

Lemma 5.8. *Let φ be a ∂ and $\bar\partial$-closed C^∞ differential form on $\widetilde{\mathcal{X}}$. Let $(F, \|\cdot\|)$ be a holomorphic Hermitian line bundle on $\widetilde{\mathcal{X}}$. Let s be a holomorphic section of F with $\mathrm{div}(s) \subset \bigcup_{b\in\mathcal{D}} \widetilde{X}_b$. Then the following identity of functions on $\mathcal{U}$ holds*

$$
\widetilde{\pi}_*((\log\|s\|^2)\,\varphi)^{(0,0)}|_{\mathcal{U}} \equiv_{\mathcal{B}} \left(\int_{\mathrm{div}(s)\cap\widetilde{X}_0} \varphi\right) \log|t|^2.
$$

In particular,

$$
\widetilde{\pi}_*(q^*(\log\|d\pi\|^2)\,\varphi)^{(0,0)}|_{\mathcal{U}} \equiv_{\mathcal{B}} \left(\int_{E_0} \varphi\right) \log|t|^2.
$$

Proof. See [61, Lemma 4.4 and Cor. 4.6] □

Since $\sum_{j=0}^{p}(-1)^{p-j}\widetilde{\mu}^*\mathrm{Td}(U, g_U)\,\widetilde{\nu}^*\{\frac{e^{(p-j)c_1(L,g_L)}-1}{c_1(L,g_L)}\}q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}, g_{\mathcal{X}})$ is a C^∞ differential form on $\widetilde{X}$ and since $\widetilde{\nu}^* c_1(L) = -\widetilde{\mu}^* c_1(H)$ in $H^2(\widetilde{\pi}^{-1}(\mathcal{U}), \mathbb{Z})$, Proposition 5.6 follows from (5.7) and Lemma 5.8. □

5.7. **Proof of Proposition 5.7**

We need the following result:

Theorem 5.9. *Let $\xi \to \mathcal{X}$ be a holomorphic vector bundle on $\mathcal{X}$ equipped with a Hermitian metric h_ξ. Let $\lambda(\xi) = \det R\pi_*\xi$ be the determinant of the cohomologies of ξ equipped with the Quillen metric $\|\cdot\|^2_{\lambda(\xi),Q}$ with respect to $g_{\mathcal{X}/S}$ and ξ. Let s be a nowhere vanishing holomorphic section of $\lambda(\xi)|_{\mathcal{U}}$. Then*

$$\log \|s\|^2_{Q,\lambda(\xi)} \equiv_{\mathcal{B}} \left(\int_{E_0} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^*\mathrm{ch}(\xi) \right) \log |t|^2.$$

Proof. See [61, Th. 1.1]. □

Let $\sigma_{(p,j)}$ be a nowhere vanishing C^∞ section of $\lambda(\Omega^j_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-j)})|_{\mathcal{U}}$. Then

$$\sigma_p := \otimes_{j=0}^p \sigma_{(p,j)}^{(-1)^{p-j}}$$

is a nowhere vanishing C^∞ section of $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}$. Since $\pi^*\Omega^1_S$ is trivial near E_0 and since

$$\log \|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S} = \sum_{j=0}^p (-1)^{p-j} \log \|\cdot\|^2_{\lambda(\Omega^j_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-j)}),Q,\pi^*k_S},$$

we deduce from Theorem 5.9 that

$$\begin{aligned}
&\log \|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,\pi^*k_S}|_{\mathcal{U}} \\
&= \sum_{j=0}^p (-1)^{p-j} \log \|\sigma_{(p,j)}\|^2_{\lambda(\Omega^j_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-j)}),Q,\pi^*k_S}|_{\mathcal{U}} \\
&\equiv_{\mathcal{B}} \sum_{j=0}^p (-1)^{p-j} \left(\int_{E_0} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}} \otimes (\pi^*\Omega^1_S)^{\otimes(p-j)}) \right) \log |t|^2 \\
&\equiv_{\mathcal{B}} \left(\int_{E_0} \sum_{j=0}^p (-1)^{p-j} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.
\end{aligned}$$

This completes the proof of Proposition 5.7. □

5.8. **An extension of Theorem 5.4**

Let $h_{\pi^{-1}(\mathcal{U})}$ be a Kähler metric on $\pi^{-1}(\mathcal{U})$, and let $h_{\mathcal{X}/S}$ be the Hermitian metric on $T\mathcal{X}/S$ induced from $h_{\pi^{-1}(\mathcal{U})}$. We do not assume that $h_{\pi^{-1}(\mathcal{U})}$ extends to a Kähler metric on $\mathcal{X}$.

Theorem 5.10. *Let σ_p be a nowhere vanishing C^∞ section of the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}$. Then*

$$\begin{aligned}
&\log \|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}}|_{\mathcal{U}} \equiv_{\mathcal{B}} \\
&\left(\int_{E_0} \sum_{j=0}^p (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \right) \log |t|^2.
\end{aligned}$$

Proof. By the anomaly formula [11, Ths. 0.2 and 0.3], we have on $\mathcal{U}^o$

$$
\begin{aligned}
&\log\left(\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}}/\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}\right)\\
&=\sum_q(-1)^q q\,\pi_*\left(\widetilde{\mathrm{Td}}(T\mathcal{X}/S;\,g_{\mathcal{X}/S},h_{\mathcal{X}/S})\,\mathrm{ch}(\Omega^q_{\mathcal{X}/S},h_{\Omega^q_{\mathcal{X}/S}})\right)^{(0,0)}\\
&\quad+\sum_q(-1)^q q\,\pi_*\left(\mathrm{Td}(T\mathcal{X}/S,g_{\mathcal{X}/S})\,\widetilde{\mathrm{ch}}(\Omega^q_{\mathcal{X}/S};\,g_{\Omega^q_{\mathcal{X}/S}},h_{\Omega^q_{\mathcal{X}/S}})\right)^{(0,0)}.
\end{aligned}
\tag{5.8}
$$

Let h_U be the Hermitian metric on U induced from $(\Pi^\vee)^*h_{\pi^{-1}(\mathcal{U})}$. Let $h_{\Omega^1_{\mathcal{X}}}$ be the Hermitian metric on $\Omega^1_{\mathcal{X}}|_{\pi^{-1}(\mathcal{U})}$ induced from $h_{\pi^{-1}(\mathcal{U})}$. Let $h_{\Omega^q_{\mathcal{X}/S}}$ be the Hermitian metric on $\Omega^q_{\mathcal{X}/S}$ induced from $h_{\Omega^1_{\mathcal{X}}}$. Let $h_{\wedge^q Q}$ be the Hermitian metric on $\wedge^q Q$ induced from $\Pi^*h_{\Omega^1_{\mathcal{X}}}$. Then we have the following isomorphisms of holomorphic Hermitian vector bundles over $\mathcal{X}\setminus\Sigma_\pi$:

$$
(T\mathcal{X}/S,h_{\mathcal{X}/S})=\mu^*(U,h_U),\qquad(\Omega^q_{\mathcal{X}/S},h_{\Omega^q_{\mathcal{X}/S}})=\nu^*(\wedge^q Q,h_{\wedge^q Q}).
\tag{5.9}
$$

By (5.1), (5.8), (5.9), we get

$$
\begin{aligned}
&\log\left(\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}}/\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}\right)\\
&=\sum_q(-1)^q q\,\widetilde{\pi}_*\left(\widetilde{\mu}^*\widetilde{\mathrm{Td}}(U;\,g_U,h_U)\,\widetilde{\nu}^*\mathrm{ch}(\wedge^q Q,h_{\wedge^q Q})\right)^{(0,0)}\\
&\quad+\sum_q(-1)^q q\,\widetilde{\pi}_*\left(\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\widetilde{\mathrm{ch}}(\wedge^q Q;\,g_{\wedge^q Q},h_{\wedge^q Q})\right)^{(0,0)}\equiv_{\mathcal{B}}0.
\end{aligned}
\tag{5.10}
$$

Here the right hand side of (5.10) lies in $\mathcal{B}(\mathcal{U})$ by [3, Th. 4bis], because

$$
\widetilde{\mu}^*\widetilde{\mathrm{Td}}(U;h_U,g_U)\,\widetilde{\nu}^*\mathrm{ch}(\wedge^q Q,h_{\wedge^q Q}),\qquad\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\widetilde{\mathrm{ch}}(\wedge^q Q;h_{\wedge^q Q},g_{\wedge^q Q})
$$

are C^∞ differential forms on $\widetilde{\pi}^{-1}(\mathcal{U})$. The result follows from Th. 5.4 and (5.10). $\square$

5.9. The case of ODP

In Subsection 5.9, we assume that $\Sigma_\pi\cap X_0$ consists of non-degenerate critical points. Hence $\mathrm{Sing}(X_0)$ consists of ODP's. For $y\in\mathcal{X}$, let $\mathfrak{m}_y$ be the maximal ideal of the local ring $\mathcal{O}_{\mathcal{X},y}$. Then there exists a neighborhood of X_0 in $\mathcal{X}$ on which $\mathcal{I}_{\Sigma_\pi}=\oplus_{y\in\mathrm{Sing}(X_0)}\mathfrak{m}_y$. Let $q\colon\widetilde{\mathcal{X}}\to\mathcal{X}$ be the blowing-up of the discrete set $\Sigma_\pi\cap X_0$, and set $E_y:=q^{-1}(y)$ for $y\in\mathrm{Sing}(X_0)$. Then $E_0=\amalg_{y\in\mathrm{Sing}(X_0)}E_y$ and $E_y\cong\mathbb{P}^n$.

Since Σ_π is discrete, we may identify $\mathbb{P}(\Omega^1_{\mathcal{X}})$ and $\mathbb{P}(T\mathcal{X})$ with the trivial projective bundle on a neighborhood of $\Sigma_\pi\cap X_0$ by fixing a system of coordinates near $\Sigma_\pi\cap X_0$. Under this trivialization, we consider the Gauss maps ν and μ only on a small neighborhood of $\Sigma_\pi\cap X_0$. Then we have the following on a neighborhood of each $y\in\Sigma_\pi\cap X_0$:

$$
\mu(z)=\nu(z)=\left(\frac{\partial\pi}{\partial z_0}(z):\cdots:\frac{\partial\pi}{\partial z_n}(z)\right).
$$

Since π is non-degenerate at every $y\in\Sigma_\pi\cap X_0$, we may assume by Morse's lemma that $\pi(z)=z_0^2+\cdots+z_n^2$ near $\Sigma_\pi\cap X_0$. Hence the composition $\nu\circ q\colon\widetilde{\mathcal{X}}\setminus E_0\to\mathbb{P}^n$ extends to a holomorphic map $\widetilde{\nu}:=\nu\circ q\colon\widetilde{\mathcal{X}}\to\mathbb{P}^n$ such that

$$
\widetilde{\nu}|_E=\widetilde{\mu}|_E=\mathrm{id}_E.
$$

For $n \in \mathbb{N}$ and $0 \leq p \leq n$, set

$$\delta(n,p) := \sum_{j=0}^{p} (-1)^j \binom{n+1}{j} \frac{(p-j+1)^{n+2} - (p-j)^{n+2}}{(n+2)!}.$$

For a formal power series $f(x) \in \mathbb{C}[[x]]$, we define $f(x)|_{x^m}$ to be the coefficient of x^m of $f(x)$. Recall that the metric $h_{\mathcal{X}/S}$ is defined only on $T\mathcal{X}/S|_{\pi^{-1}(\mathcal{U})\setminus\Sigma_\pi}$.

Theorem 5.11. *Let σ_p be a nowhere vanishing C^∞ section of the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}$. Then the following identity of functions on $\mathcal{U}$ holds*

$$(-1)^p \log \|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}} \equiv_{\mathcal{B}} (-1)^n \delta(n,p)\, \#\mathrm{Sing}(X_0)\, \log |t|^2.$$

Proof. In Theorem 5.10, we can identify U (resp. L) with the universal hyperplane bundle (resp. tautological line bundle) on $\mathbb{P}^n$. Then $H = L^{-1}$. Set $x := c_1(H)$. Hence $\int_{\mathbb{P}^n} x^n = 1$. From the exact sequence $0 \to U \to \mathbb{C}^{n+1} \to H \to 0$, we get $\mathrm{Td}(U) = \mathrm{Td}^{-1}(x) = (1-e^{-x})/x$. Since $q(E_0)$ consists of a point, we get $q^*\Omega^j_{\mathcal{X}}|_{E_0} = \mathbb{C}^{\oplus\binom{n+1}{p}}$. By substituting this and the equation $q^*\mathrm{ch}(\Omega^j_{\mathcal{X}})|_{E_0} = \binom{n+1}{p}$ into the formula in Theorem 5.10, we get

$$\begin{aligned}
&\int_E \sum_{j=0}^{p} (-1)^{p-j}\, \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} q^*\mathrm{ch}(\Omega^j_{\mathcal{X}}) \\
&= \#\mathrm{Sing}(X_0) \sum_{j=0}^{p} (-1)^{p-j} \frac{1}{\mathrm{Td}(x)} \cdot \frac{\mathrm{Td}(x) - e^{-(p-j)x}}{x} \cdot \binom{n+1}{j} \Bigg|_{x^n} \\
&= \#\mathrm{Sing}(X_0) \sum_{j=0}^{p} (-1)^{p-j} \binom{n+1}{j} \left\{ \frac{(e^{-x}-1)e^{-(p-j)x}}{x^2} + \frac{1}{x} \right\} \Bigg|_{x^n} \\
&= \#\mathrm{Sing}(X_0) \sum_{j=0}^{p} (-1)^{p-j} \binom{n+1}{j} \{e^{-(p-j+1)x} - e^{-(p-j)x}\}|_{x^{n+2}} \\
&= (-1)^{n-p}\, \delta(n,p)\, \#\mathrm{Sing}(X_0).
\end{aligned} \tag{5.11}$$

The result follows from Theorem 5.4 and (5.11). □

Lemma 5.12. *The following identities hold:*

$$\delta(3,p) + \delta(3,3-p) = 1 \quad (0 \leq p \leq 3), \qquad \sum_{p=0}^{3} p\, \delta(3,p) = \frac{19}{4}.$$

Proof. By the definition of $\delta(n,p)$, we get

$$\delta(3,0) = \frac{1}{120}, \quad \delta(3,1) = \frac{27}{120}, \quad \delta(3,2) = \frac{93}{120}, \quad \delta(3,3) = \frac{119}{120},$$

which yields the result. □

Set

$$\sigma := \otimes_{p=0}^{n} \sigma_p^{(-1)^p p}.$$

Then σ is a nowhere vanishing C^∞ section of $\lambda(\Omega^\bullet_{\mathcal{X}/S})$ near $\mathcal{D}$.

Theorem 5.13. *When $n = 3$,*

$$\log \|\sigma(t)\|^2_{\lambda(\Omega^\bullet_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}} \equiv_{\mathcal{B}} -\frac{19}{4}\,\#\mathrm{Sing}(X_0)\,\log|t|^2.$$

Proof. By Theorem 5.11, we get

$$\begin{aligned}\log\|\sigma\|^2_{\lambda(\Omega^\bullet_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}|_{\mathcal{U}} &= \sum_{p=0}^{3}(-1)^p p\,\log\|\sigma_p\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}|_{\mathcal{U}}\\ &\equiv_{\mathcal{B}} (-1)^3\sum_{p=0}^{3} p\,\delta(3,p)\,\#\mathrm{Sing}(X_0)\,\log|t|^2.\end{aligned}$$

This, together with the second identity of Lemma 5.12, yields the result. □

Remark 5.14. In our subsequent paper [18], we shall determine the behavior of $\log\|\sigma(t)\|^2_{\lambda(\Omega^\bullet_{\mathcal{X}/S}),Q,h_{\mathcal{X}/S}}$ as $t \to 0$ for arbitrary relative dimension n.

6. **The cotangent sheaf of the Kuranishi space**

Let X be a smoothable Calabi-Yau n-fold with only one ODP as its singular set. Let $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X with discriminant locus $\mathfrak{D}$. Then $\mathfrak{X}$, $\mathrm{Def}(X)$, and $\mathfrak{D}$ are smooth by Lemmas 2.3 and 2.7.

Lemma 6.1. *The dualizing sheaf $K_{\mathfrak{X}}$ of $\mathfrak{X}$ is trivial. In particular, the relative dualizing sheaf $K_{\mathfrak{X}/\mathrm{Def}(X)} = K_{\mathfrak{X}} \otimes (\mathfrak{p}^* K_{\mathrm{Def}(X)})^{-1}$ is trivial.*

Proof. By the same argument as in [60, p.68 l.25-l.28], we see that $K_{\mathfrak{X}}|_{X_s} \cong \mathcal{O}_{X_s}$ for all $s \in \mathrm{Def}(X)$. Since $\mathrm{Def}(X) \cong \Delta^{N+1}$, we get the triviality of $K_{\mathfrak{X}}$ by the same argument as in [60, p.68 l.29-l.33]. □

Recall that the Kodaira-Spencer isomorphism

$$\rho_{\mathrm{Def}(X)\setminus\mathfrak{D}}\colon \Theta_{\mathrm{Def}(X)\setminus\mathfrak{D}} \to R^1\mathfrak{p}_*\Theta_{\mathfrak{X}/\mathrm{Def}(X)}|_{\mathrm{Def}(X)\setminus\mathfrak{D}}$$

was defined in Subsection 4.2. By considering the dual of $\rho_{\mathrm{Def}(X)\setminus\mathfrak{D}}$, the relative Serre duality induces an isomorphism of $\mathcal{O}_{\mathrm{Def}(X)}$-modules on $\mathrm{Def}(X)\setminus\mathfrak{D}$:

$$\rho^\vee_{\mathrm{Def}(X)\setminus\mathfrak{D}}\colon R^{n-1}\mathfrak{p}_*(\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)} \otimes K_{\mathfrak{X}/\mathrm{Def}(X)})|_{\mathrm{Def}(X)\setminus\mathfrak{D}} \cong \Omega^1_{\mathrm{Def}(X)}|_{\mathrm{Def}(X)\setminus\mathfrak{D}}.$$

Theorem 6.2. *The isomomorphism $\rho^\vee_{\mathrm{Def}(X)\setminus\mathfrak{D}}$ extends to an isomorphism*

$$\rho^\vee_{\mathrm{Def}(X)}\colon R^{n-1}\mathfrak{p}_*(\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)} \otimes K_{\mathfrak{X}/\mathrm{Def}(X)}) \cong \Omega^1_{\mathrm{Def}(X)}.$$

of $\mathcal{O}_{\mathrm{Def}(X)}$-modules over $\mathrm{Def}(X)$.

The isomorphism $\rho^\vee_{\mathrm{Def}(X)}$ is again called the Kodaira-Spencer isomorphism. Before proving Theorem 6.2, we first prove an intermediate result in the next subsection.

6.1. Blowing-up and the regularity of differential forms

Set

$$\widetilde{\Delta^{n+1}} = \{(z,[\zeta]) \in \Delta^{n+1} \times \mathbb{P}^n;\ z_i\zeta_j - z_j\zeta_i = 0 \quad 0 \le i,j \le n\}, \qquad q := \mathrm{pr}_1.$$

Then $q\colon \widetilde{\Delta^{n+1}} \to \Delta^{n+1}$ is the blowing-up at the origin. Set $E := q^{-1}(0)$ and

$$U_i := \{(z,[\zeta]) \in \widetilde{\Delta^{n+1}};\ \zeta_i \neq 0\}, \qquad O_i := \{z \in \Delta^{n+1};\ z_i \neq 0\},$$
$$W_i := \{(\zeta_0,\ldots,\zeta_{i-1},z_i,\zeta_{i+1},\ldots,\zeta_n) \in \mathbb{C}^{n+1};\ |z_i|<1, \quad |z_i\zeta_j|<1 \quad (j \neq i)\}.$$

Then $U_i \cong W_i \subset \mathbb{C}^{n+1}$ via the map

$$W_i \ni (\zeta_0,\ldots,\zeta_{i-1},z_i,\zeta_{i+1},\ldots,\zeta_n)$$
$$\to ((z_i\zeta_0,\ldots,z_i\zeta_{i-1},z_i,z_i\zeta_{i+1},\ldots,z_i\zeta_n),[\zeta_0:\cdots:\zeta_{i-1}:1:\zeta_{i+1}:\cdots:\zeta_n]) \in U_i.$$

By construction, we have $\widetilde{\Delta}^{n+1} = \bigcup_{i=0}^n U_i$ and

$$E \cap U_i \cong \{(\zeta_0,\ldots,\zeta_{i-1},z_i,\zeta_{i+1},\ldots,\zeta_n) \in W_i;\ z_i = 0\}, \qquad q(U_i) \supset O_i.$$

Let ω_{ij} be the C^∞ $(n,0)$-form on O_i defined by

$$\omega_{ij} := \frac{|z_j|^2}{|z_0|^2+\cdots+|z_n|^2} \cdot \frac{dz_0 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n}{z_i^{n-2} z_j}.$$

Lemma 6.3. *For all $0 \le i,j \le n$, the C^∞ $(n,0)$-form $q^*\omega_{ij}$ on $q^{-1}(O_i) = U_i \setminus E$ extends to a C^∞ $(n,0)$-form on U_i and satisfies $q^*\omega_{ij}|_{E\cap U_i} = 0$.*

Proof. Since $q|_{W_i}(\zeta_0,\ldots,\zeta_{i-1},z_i,\zeta_{i+1},\ldots,\zeta_n) = (z_i\zeta_0,\ldots,z_i\zeta_{i-1},z_i,z_i\zeta_{i+1},\ldots,z_i\zeta_n)$ under the identification $U_i \cong W_i$, we get the following two formulas:

$$q^*\left(\frac{|z_j|^2}{|z_0|^2+\cdots+|z_n|^2}\right) = \begin{cases} |\zeta_j|^2(1+\|\zeta\|^2)^{-1} & (j \neq i) \\ (1+\|\zeta\|^2)^{-1} & (j = i), \end{cases}$$

$$\begin{aligned} &q^*\left(z_i^{-(n-1)} dz_0 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n\right) \\ &= z_i^{-(n-1)} d(z_i\zeta_0) \wedge \cdots \wedge d(z_i\zeta_{i-1}) \wedge d(z_i\zeta_{i+1}) \wedge \cdots \wedge d(z_i\zeta_n) \\ &= z_i\, d\zeta_0 \wedge \cdots \widehat{d\zeta_i} \cdots \wedge d\zeta_n + dz_i \wedge \sum_{j<i} (-1)^{j-1} d\zeta_0 \wedge \cdots \widehat{d\zeta_j} \cdots \widehat{d\zeta_i} \cdots \wedge d\zeta_n \\ &\quad + dz_i \wedge \sum_{j>i} (-1)^j d\zeta_0 \wedge \cdots \widehat{d\zeta_i} \cdots \widehat{d\zeta_j} \cdots \wedge d\zeta_n \in A^{n,0}(U_i), \end{aligned}$$

which yields that $q^*\omega_{ii} \in A^{n,0}(U_i)$ and $q^*\omega_{ii}|_{E\cap U_i} = 0$. Since $q^*\omega_{ij} = \frac{\bar{\zeta}_j}{1+\|\zeta\|^2}\, q^*\omega_{ii}$ when $j \neq i$, the assertion for $q^*\omega_{ij}$ $(i \neq j)$ follows from the assertion for $q^*\omega_{ii}$. □

6.2. Proof of Theorem 6.2

For simplicity, we set

$$\mathcal{X} := \mathfrak{X}, \quad S := \mathrm{Def}(X), \quad \pi := \mathfrak{p}, \quad 0 := [X], \quad X_0 := X, \quad N+1 = \dim S.$$

Hence $(S,0) \cong (\Delta^{N+1},0)$ and $\pi\colon (\mathcal{X},X_0) \to (S,0)$ is the Kuranishi family of X_0.

Let $s = (s_0,\ldots,s_N)$ be a system of coordinates of S such that $\mathfrak{D} = \mathrm{div}(s_0)$. We set $s' = (s_1,\ldots,s_N)$. Then $\partial/\partial s_\alpha$ is a nowhere vanishing holomorphic vector field on S for $0 \le \alpha \le N$.

(Step 1) The Kodaira-Spencer isomorphism $\rho_{S\setminus\mathfrak{D}}\colon \Theta_{S\setminus\mathfrak{D}} \to R^1\pi_*\Theta_{\mathcal{X}/S}|_{S\setminus\mathfrak{D}}$ yields holomorphic sections $\rho(\partial/\partial s_\alpha) \in H^0(S\setminus\mathfrak{D}, R^1\pi_*\Theta_{\mathcal{X}/S})$. Let $\langle\cdot,\rangle_s$ be the Yoneda product between $H^{n-1}(X_s,\Omega^1_{X_s}\otimes K_{X_s})$ and $\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(\Omega^1_{X_s}\otimes K_{X_s},K_{X_s})$.

Since $h^{n-1}(X_s,\Omega^1_{X_s})=N+1$, there exist $\phi_0,\dots,\phi_N\in H^{n-1}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})$ such that
(i) $\{\phi_0,\dots,\phi_N\}$ is a basis of $R^{n-1}\pi_*(\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})$ as a free $\mathcal{O}_S$-module;
(ii) $\{\phi_0|_{X_s},\dots,\phi_N|_{X_s}\}$ is a basis of $H^{n-1}(X_s,\Omega^1_{X_s}\otimes K_{X_s})$ for all $s\in S$;
(iii) $\langle\phi_\alpha|_{X_0},\rho_0(\partial/\partial s_\beta)\rangle_0=\delta_{\alpha\beta}$ for $0\leq\alpha,\beta\leq N$.

Let $\rho_s^\vee\colon H^{n-1}(X_s,\Omega^1_{X_s}\otimes K_{X_s})\to\Omega^1_{S,s}$ be the dual of the Kodaira-Spencer map. For $s\in S$, set

$$g_{\alpha\beta}(s):=\langle\phi_\alpha|_{X_s},\rho_s(\partial/\partial s_\beta)\rangle_s=\langle\langle\rho_s^\vee(\phi_\alpha|_{X_s}),\partial/\partial s_\beta\rangle\rangle,$$

where $\langle\langle\cdot,\cdot\rangle\rangle\colon\Omega^1_{S,s}\times TS_s\to\mathbb{C}$ is the natural pairing. Then $g_{\alpha\beta}$ is a function on S, which is holomorphic on $S\setminus\mathfrak{D}$ but which may not be continuous on S, such that

$$g_{\alpha\beta}(0)=\delta_{\alpha\beta}.$$

It suffices to prove $g_{\alpha\beta}\in C^0(S)$; if it is the case, $(g_{\alpha\beta}(s))$ is a family of invertible matrices depending holomorphically on $s\in S$, so that $R^{n-1}\pi_*(\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})$ is the holomorphic dual bundle of Θ_S via the extension of $\rho^\vee_{S\setminus\mathfrak{D}}$.

(Step 2) Let $\mathcal{A}_\mathcal{X}$ be the sheaf of germs of C^∞ functions on $\mathcal{X}$, and let $\mathcal{A}^{p,q}_\mathcal{X}$ be the sheaf of germs of C^∞ (p,q)-forms on $\mathcal{X}$. Set

$$A^{p,q}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S}):=\Gamma(\mathcal{X},\mathcal{A}^{p,q}_\mathcal{X}\otimes_{\mathcal{O}_\mathcal{X}}\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S}).$$

Then $A^{p,q}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})$ is the vector space of C^∞ (p,q)-forms on $\mathcal{X}$ with values in $\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S}$. By Malgrange [38, pp. 88, Cor. 1.12], $\mathcal{O}_\mathcal{X}$ is a flat $\mathcal{A}_\mathcal{X}$-module. Hence we have the Dolbeault isomorphism [1, Chap. VII, Prop. 4.5]

$$\begin{aligned}&H^{n-1}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})\\&=\frac{\ker\{\bar\partial\colon A^{0,n-1}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})\to A^{0,n}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})\}}{\mathrm{Im}\{\bar\partial\colon A^{0,n-2}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})\to A^{0,n-1}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})\}}.\end{aligned}$$

Let $\Phi_\alpha\in A^{0,n-1}(\mathcal{X},\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S})$ be a $\bar\partial$-closed differential form representing ϕ_α, i.e., $\phi_\alpha=[\Phi_\alpha]$.

(Step 3) To study the behavior of $g_{\alpha\beta}(s)$ near $\mathfrak{D}$, we compute a representative of the Kodaira-Spencer classes $\rho(\partial/\partial s_\alpha)$ in the Dolbeault cohomology.

Near the critical locus $\Sigma_\pi\subset\mathcal{X}$, there is a neighborhood $V\cong\Delta^{n+1}\times\Delta^N$ of Σ_π in $\mathcal{X}$ such that $\pi(z_0,\dots,z_n,s')=(z_0^2+\cdots+z_n^2,s_1,\dots,s_N)$. Hence we have $\Sigma_\pi\cap V=\{0\}\times\Delta^N$. For $i=0,1,\dots,n$, we set

$$V_i:=\Delta^{i-1}\times\Delta^*\times\Delta^{n-i}\times\Delta^N=\{(z,s')\in\Delta^{n+1}\times\Delta^N;\,z_i\neq0\}.$$

Then $\{V_i\}_i$ is an open covering of $V\setminus\Sigma_\pi$, i.e., $V\setminus\Sigma_\pi=\bigcup_{i=0}^n V_i$. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be an open covering of $\overline{\mathcal{X}\setminus V}$ such that $V_\lambda\cong\Delta^n\times\Delta^{N+1}$ and $\pi|_{V_\lambda}=\mathrm{pr}_2$. Then $\mathfrak{V}:=\{V_i\}_i\cup\{V_\lambda\}_\lambda$ is an open covering of $\mathcal{X}\setminus\Sigma_\pi$.

First let us construct a representative of the Kodaira-Spencer class $\rho(\partial/\partial s_\alpha)$ in the Cech cohomology with respect to the covering $\mathfrak{V}$.

On V_i, set

$$v_0^{(i)}:=\frac{1}{2z_i}\frac{\partial}{\partial z_i},\qquad v_\alpha^{(i)}=\frac{\partial}{\partial s_\alpha}\quad(\alpha=1,\dots,N).$$

Then $v_0^{(i)},\dots v_N^{(i)}\in H^0(V_i,\Theta_\mathcal{X})$ and $\pi_*(v_\alpha^{(i)})=\frac{\partial}{\partial s_\alpha}$ $(\alpha=0,\dots,N)$. We also fix a holomorphic vector field $v_\alpha^{(\lambda)}$ such that $v_\alpha^{(\lambda)}=\partial/\partial s_\alpha$ on every V_λ. We get in Cech

cohomology

$$\rho\left(\frac{\partial}{\partial s_\alpha}\right)=\{(v_\alpha^{(\mu)}-v_\alpha^{(\nu)})|_{V_\mu\cap V_\nu}\}_{V_\mu,V_\nu\in\mathfrak{V}}\in H^1(\mathcal{X}\setminus\Sigma_\pi,\Theta_{\mathcal{X}/S};\mathfrak{V}).$$

Let $\{\chi_i\}_i\cup\{\chi_\lambda\}_\lambda$ be a partition of unity of $\mathcal{X}\setminus\Sigma_\pi$ subject to the covering $\mathfrak{V}$ such that on V_i,

$$\chi_i(z)=\frac{|z_i|^2}{|z_0|^2+\cdots+|z_n|^2},\qquad i=0,\dots,n.$$

Then the following differential form $\xi_\alpha\in A^{0,1}(\mathcal{X}\setminus\Sigma_\pi,\Theta_{\mathcal{X}/S})$ represents $\rho(\partial/\partial s_\alpha)$:

$$\xi_\alpha|_{V_\nu}:=\sum_\mu\bar\partial\chi_\mu\otimes(v_\alpha^{(\mu)}-v_\alpha^{(\nu)}),\qquad\rho_s\left(\frac{\partial}{\partial s_\alpha}\right)=[\xi_\alpha|_{X_s}]\quad(s\in S\setminus\mathfrak{D}).$$

In particular, we get on $V\setminus\Sigma_\pi$

$$\xi_0|_{V\setminus\Sigma_\pi}=\sum_{i=0}^n\bar\partial\chi_i\otimes\frac{1}{2z_i}\frac{\partial}{\partial z_i},\qquad\xi_\alpha|_{V\setminus\Sigma_\pi}=0\quad(\alpha=1,\dots,N).$$

(Step 4) Let us study the behavior of $g_{\alpha\beta}|_{S\setminus\mathfrak{D}}(s)$ as $s\to\mathfrak{D}$. Let $\varrho(z)\in C_0^\infty(\Delta^{n+1})$ be a cut-off function with $\varrho\equiv1$ near $0\in\Delta^{n+1}$. Recall that $\iota(\cdot)$ denotes the interior product. There exists $h_{\alpha\beta}(s)\in C^\infty(S)$ such that for $s\in S\setminus\mathfrak{D}$,

$$g_{\alpha\beta}(s)=\langle\phi_\alpha|_{X_s},\rho_s(\partial/\partial s_\beta)\rangle_s=\int_{X_s}\iota(\xi_\beta)\Phi_\alpha=\int_{X_s\cap V}\varrho(z)\cdot\iota(\xi_\beta)\Phi_\alpha+h_{\alpha\beta}(s).$$

Since $\xi_\beta\equiv0$ on $V\setminus\Sigma_\pi$ for $\beta\neq0$, $g_{\alpha\beta}|_{S\setminus\mathfrak{D}}(s)$ extends to a C^∞ function on S if $\beta\neq0$. Let us prove that $g_{\alpha0}|_{S\setminus\mathfrak{D}}$ extends to a continuous function on S.

Since Φ_α is a $(0,n-1)$-form on $\mathcal{X}$ with values in $\Omega^1_{\mathcal{X}/S}\otimes K_{\mathcal{X}/S}$, we can write

$$\Phi_\alpha|_V=\sum_{i=0}^n\theta_\alpha^i(z,s)\,[dz_i]\otimes\eta,$$

with $[dz_i]=dz_i\mod(\pi^*ds_0,\dots,\pi^*ds_N)$, $\theta^i_\alpha\in A^{0,n-1}(V)$, and

$$\eta|_{V_i}:=(-1)^{i-1}\left.\frac{dz_0\wedge\cdots\wedge dz_{i-1}\wedge dz_{i+1}\wedge\cdots\wedge dz_n}{2z_i}\right|_{V_i}=\operatorname{Res}\left.\left(\frac{dz_0\wedge\cdots\wedge dz_n}{z_0^2+\cdots+z_n^2}\right)\right|_{V_i}.$$

Hence we have the following formula on V_i

$$\begin{aligned}\iota(\xi_0)\Phi_\alpha|_{V_i}&=\iota(\sum_{j=0}^n\bar\partial\chi_j\otimes\frac{1}{2z_j}\frac{\partial}{\partial z_j})\sum_{k=0}^n\theta_\alpha^k\,[dz_k]\otimes\eta|_{V_i}\\&=\frac12\sum_{j=0}^n\frac{\bar\partial\chi_j\wedge\theta_\alpha^j}{z_j}\wedge\eta|_{V_i}=\frac14\sum_{i=0}^n(-1)^{n+i}\,z_i^{n-3}\,\theta_\alpha^j\wedge\bar\partial\omega_{ij},\end{aligned}\tag{6.1}$$

where we used the following relations to get the second equality:

$$\iota(\sum_{j=0}^n\bar\partial\chi_j\otimes\frac{1}{2z_j}\frac{\partial}{\partial z_j})\,\pi^*ds_k=0,\qquad k=0,\dots,N.$$

Let $q\colon\widetilde{\mathcal{X}}\to\mathcal{X}$ be the blowing-up along the submanifold $\Sigma_\pi\subset V$ with exceptional divisor $E:=q^{-1}(\Sigma_\pi)=\mathbb{P}(N_{\Sigma_\pi/\mathcal{X}})$. Then $q|_E\colon\mathbb{P}(N_{\Sigma_\pi/V})\to\Sigma_\pi$ is the standard projection. Since $n\geq3$ and since $\{U_i\times\Delta^N\}_i$ is an open covering of $\widetilde{V}:=q^{-1}(V)$, we deduce from Lemma 6.3 and (6.1) that $q^*(\iota(\xi_0)\Phi_\alpha)\in A^{(n,n)}(\widetilde{\mathcal{X}})$.

Set $\widetilde{\pi} = \pi \circ q$. By King [30, Th. 3.3.2], we have $\widetilde{\pi}_* q^*(\iota(\xi_\alpha)\Phi_\alpha) \in C^0(S)$. Since

$$g_{\alpha 0}|_{S \setminus \mathfrak{D}} = \pi_*(\iota(\xi_0)\Phi_\alpha) = \widetilde{\pi}_* q^*(\iota(\xi_0)\Phi_\alpha),$$

$g_{\alpha 0}|_{S\setminus\mathfrak{D}}$ extends to a continuous function on S.

(Step 5) Let $s_0 \in \mathfrak{D}$. We must prove $\lim_{S\setminus\mathfrak{D}\ni s\to s_0} g_{\alpha\beta}|_{S\setminus\mathfrak{D}}(s) = g_{\alpha\beta}(s_0)$. Let Y_{s_0} be the proper transform of X_{s_0}. Since $q^{-1}(X_{s_0}) = Y_{s_0} \cup E$ and since $g_{\alpha\beta}|_{S\setminus\mathfrak{D}}$ extends to a continuous function on S, we get

$$\lim_{s\to s_0} g_{\alpha\beta}|_{S\setminus\mathfrak{D}}(s) = \int_{q^{-1}(X_{s_0})} q^*(\iota(\xi_\beta)\Phi_\alpha) = \int_{Y_{s_0}} q^*(\iota(\xi_\beta)\Phi_\alpha) + \int_E q^*(\iota(\xi_\beta)\Phi_\alpha).$$

Since $q^*(\iota(\xi_\beta)\Phi_\alpha)|_E = 0$ by Lemma 6.3 and (6.1), we get

$$\lim_{s\to s_0} g_{\alpha\beta}|_{S\setminus\mathfrak{D}}(s) = \int_{Y_{s_0}} q^*(\iota(\xi_\beta)\Phi_\alpha) = \int_{(X_{s_0})_{\mathrm{reg}}} \iota(\xi_\beta)\Phi_\alpha = \langle \phi_\alpha|_{X_{s_0}}, \rho_{s_0}(\frac{\partial}{\partial s_\beta})\rangle_{s_0} = g_{\alpha\beta}(s_0),$$

where we used Lemma 2.9 to get the third equality. This proves $g_{\alpha\beta}(s) \in C^0(S)$. This completes the proof of Theorem 6.2. $\square$

7. Behaviors of the Weil-Petersson metric and the Hodge metric

In this section, we study the boundary behavior of the Weil-Petersson metric and the Hodge metric for one-parameter families of Calabi-Yau threefolds that shall be used later. We first recall some basic notions about positive $(1,1)$-current and give two lemmas on harmonic functions on Δ^*.

7.1. Positive $(1,1)$-currents and their trivial extensions

Let u be a $(1,1)$-current on Δ. Then u is *positive* if u is real and if the inequality $u(\varphi) \geq 0$ holds for all non-negative function $\varphi \in C_0^\infty(\Delta)$. For real $(1,1)$-currents u, v on Δ, $u \geq v$ if $u - v$ is a positive $(1,1)$-current on Δ. For a divisor H on Δ, let δ_H be the current of integration over H. A real-valued function $f \in L^1_{\mathrm{loc}}(\Delta)$ is *subharmonic* if f is upper semi-continuous and if $dd^c f \geq 0$ as currents on Δ.

Let ω_{Δ^*} be the Kähler form of the Poincaré metric on Δ^*:

$$\omega_{\Delta^*} := \frac{\sqrt{-1}dt\wedge d\bar{t}}{|t|^2(-\log|t|^2)^2} = -dd^c \log(-\log|t|^2).$$

A C^∞ real $(1,1)$-form T on Δ^* has Poincaré growth if there exists $C > 0$ with

$$-C\,\omega_{\Delta^*} \leq T \leq C\,\omega_{\Delta^*}. \tag{7.1}$$

In that case, the coefficient of T lies in $L^1_{\mathrm{loc}}(\Delta)$. The $(1,1)$-current on Δ defined by

$$\widetilde{T}(\psi) := \int_\Delta \psi\, T, \qquad \psi \in C_0^\infty(\Delta)$$

is called the *trivial extension* of T from Δ^* to Δ. We have $\widetilde{\omega_{\Delta^*}} = -dd^c\log(-\log|t|^2)$ as currents on Δ.

7.2. Two lemmas on harmonic functions on Δ^*

Lemma 7.1. *Let $H(t)$ be a real-valued harmonic function on Δ^*.*
(1) *There exist $c \in \mathbb{R}$ and $F(t) \in \mathcal{O}(\Delta^*)$ with $H(t) = c\log|t|^2 + 2\mathrm{Re}\,F(t)$.*
(2) *If there exist $\gamma \in \mathbb{R}$ such that $|t|^\gamma e^{H(t)} \in L^1_{\mathrm{loc}}(\Delta)$, then $F(t) \in \mathcal{O}(\Delta)$.*
(3) *If $H(t) = O(\log(-\log|t|))$ as $t \to 0$, then $H(t)$ extends to a harmonic function on Δ.*

Proof. **(1)** Since $H(t)$ is harmonic on Δ^*, there exists $f(t) \in \mathcal{O}(\Delta^*)$ with $\partial H(t) = f(t)\,dt$. Let $f(t) = \sum_{n\in\mathbb{Z}} a_n\,t^n$ be the Laurent expansion of $f(t)$ and define the meromorphic function $F(t)$ on Δ^* by $F(t) := \sum_{n\neq -1} \frac{a_n}{n+1}\,t^{n+1}$. By the reality of $H(t)$, we get $dH(t) = a_{-1}\frac{dt}{t} + \overline{a}_{-1}\frac{d\bar{t}}{\bar{t}} + dF(t) + d\bar{F}(t)$. Integrating the both hand sides over the circle $|t| = 1/2$, we get $a_{-1} \in \mathbb{R}$ by the Stokes theorem, so that $dH(t) = a_{-1}\,d\log|t|^2 + 2d\{\mathrm{Re}\,F(t)\}$. This proves (1).
(2) By assumption, we get

$$
\int_{|t|<1/2} |t|^{\gamma+2c}\,|e^{F(t)}|^2\,\sqrt{-1}\,dt\wedge d\bar{t} < +\infty. \tag{7.2}
$$

Since $e^{F(t)}$ is holomorphic on Δ^*, we deduce from (7.2) that $e^{F(t)}$ is a meromorphic function on Δ. There exist $\nu \in \mathbb{Z}$ and a nowhere vanishing holomorphic function $\epsilon(t) \in \mathcal{O}(\Delta)$ with $e^{F(t)} = t^{\nu}\,\epsilon(t)$. Then $F'(t) = \nu\,t^{-1} + \epsilon'(t)\epsilon(t)^{-1}$. Since $F(t)$ is a meromorphic function on Δ^*, the residue of $F'(t)$ must vanish, i.e., $\nu = 0$. Thus we have proved that $F(t) = \log\epsilon(t)$ is holomorphic on Δ.
(3) Since $e^{H(t)} \in L^1_{\mathrm{loc}}(\Delta)$, $H(t) - c\,\log|t|^2$ is a harmonic function on Δ by (1), (2). Since $H(t) = O(\log(-\log|t|))$ as $t \to 0$, we get $c = 0$. This completes the proof. □

Lemma 7.2. *Let $\lambda(t)$ be a positive, locally L^m-integrable function on Δ for some $m > 0$. Let $\chi(t)$ be a function on Δ^* satisfying $\chi(t) \leq C\,(-\log|t| + 2)$, where $C \in \mathbb{R}$ is a constant. If $\log\lambda(t) + \chi(t)$ is harmonic on Δ^*, then there exists $c \in \mathbb{R}$ such that*

$$
\log\lambda(t) = c\,\log|t|^2 + O(|\chi(t)| + 1) \qquad (t \to 0).
$$

Proof. Set $H(t) := \log\lambda(t) + \chi(t)$. Since $\chi(t) \leq C\,(-\log|t| + 2)$, we get

$$
\log\lambda(t) = H(t) - \chi(t) \geq H(t) - C\,(-\log|t| + 2). \tag{7.3}
$$

Since $\lambda(t) \in L^m(\Delta(1/2))$, we get

$$
e^{-2Cm}\int_{\Delta(1/2)} |t|^{Cm} e^{m\,H(t)}\,\sqrt{-1}\,dt\wedge d\bar{t} \leq \int_{\Delta(1/2)} \lambda(t)^m\,\sqrt{-1}\,dt\wedge d\bar{t} < +\infty. \tag{7.4}
$$

By (7.4) and Lemma 7.1 (1), (2), there exists $c \in \mathbb{R}$ and $F(t) \in \mathcal{O}(\Delta)$ with

$$
H(t) = c\,\log|t|^2 + 2\,\mathrm{Re}\,F(t). \tag{7.5}
$$

Since $\log\lambda(t) = H(t) - \chi(t)$, the result follows from (7.5). □

7.3. The boundary behaviors

In Subsect. 7.3, we fix the following notation. Let $\mathcal{X}$ be a (possibly) singular complex fourfold and let $\pi\colon \mathcal{X} \to \Delta$ be a proper surjective holomorphic function. Assume that $X_t := \pi^{-1}(t)$ is a smooth Calabi-Yau threefold for $t \in \Delta^*$. We do *not* assume that the central fiber X_0 has only ODP's as its singular set. Recall that the Weil-Petersson form $\omega_{\mathrm{WP},\mathcal{X}/\Delta}$ and the Hodge form $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ for $\pi\colon \mathcal{X} \to \Delta$ were defined in Sects. 4.3 and 4.4.3, respectively.

Proposition 7.3. *There exists a positive constant C such that*

$$
0 \leq \omega_{\mathrm{WP},\mathcal{X}/\Delta} \leq C\,\omega_{\Delta^*}, \qquad 0 \leq \omega_{\mathrm{H},\mathcal{X}/\Delta} \leq C\,\omega_{\Delta^*}. \tag{7.6}
$$

In particular, the positive $(1,1)$-forms $\omega_{\mathrm{WP},\mathcal{X}/\Delta}$ and $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ on Δ^* extend trivially to closed positive $(1,1)$-currents on Δ.

Proof. We follow [37, Proof of Th. 5.1]. Since (7.6) is obvious when $\omega_{\mathrm{H},\mathcal{X}/\Delta}=0$, we assume that $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ does not vanish identically on Δ^*. Shrinking Δ if necessary, we may assume that $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ is strictly positive on Δ^*. Let $b\in\Delta^*$. Since $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ is non-degenerate at b, the deformation germ $\pi\colon(\mathcal{X},X_b)\to(\Delta,b)$ is induced from the Kuranishi family by an immersion of germs $(\Delta,b)\hookrightarrow(\mathrm{Def}(X_b),[X_b])$. Let ω_{H} be the Hodge form on $\mathrm{Def}(X_b)$. By [36, Th. 1.1.2], the holomorphic sectional curvature of $(\mathrm{Def}(X_b),\omega_{\mathrm{H}})$ is bounded from above by $\alpha:=-(5+2\sqrt{3})^{-1}$. Since $b\in\Delta^*$ is an arbitrary point, the holomorphic sectional curvature of $(\Delta^*,\omega_{\mathrm{H},\mathcal{X}/\Delta})$ is bounded from above by α (cf. e.g. [29, Prop. 2.3.9]). The second inequality of (7.6) follows from the Schwarz lemma [29, Th. 2.3.5]. The first inequality of (7.6) follows from the second one because $2\,\omega_{\mathrm{WP},\mathcal{X}/\Delta}\leq\omega_{\mathrm{H},\mathcal{X}/\Delta}$ by [36, p.107, l.17].

Since $(\Delta(r)^*,\omega_{\Delta^*})$ has finite volume when $r<1$, the positive $(1,1)$-forms $\omega_{\mathrm{WP},\mathcal{X}/\Delta}$ and $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ extend trivially to closed positive $(1,1)$-currents on Δ. □

Definition 7.4. Define $\Omega_{\mathrm{WP},\mathcal{X}/\Delta}$ and $\Omega_{\mathrm{H},\mathcal{X}/\Delta}$ as the trivial extensions of $\omega_{\mathrm{WP},\mathcal{X}/\Delta}$ and $\omega_{\mathrm{H},\mathcal{X}/\Delta}$ from Δ^* to Δ, respectively.

Lemma 7.5. *Let $A,B\in\mathbb{R}$. Let $\lambda(t)$ be a positive, locally L^m-integrable C^∞ function on Δ^* for some $m>0$ such that $-dd^c\log\lambda=A\,\omega_{\mathrm{H},\mathcal{X}/\Delta}+B\,\omega_{\mathrm{WP},\mathcal{X}/\Delta}$.*
(1) *There exists $c\in\mathbb{R}$ such that as $t\to0$,*

$$\log\lambda(t)=c\,\log|t|^2+O(\log(-\log|t|)).$$

(2) *With the same constant c as above, the following equation of currents on Δ holds:*

$$-dd^c\log\lambda=A\,\Omega_{\mathrm{H},\mathcal{X}/\Delta}+B\,\Omega_{\mathrm{WP},\mathcal{X}/\Delta}-c\,\delta_0.$$

Proof. We follow [60, Prop. 3.11]. By [49, Proof of Lemma 5.4], there exist subharmonic functions φ and θ on Δ such that the following equations of currents on Δ hold:

$$\Omega_{\mathrm{WP},\mathcal{X}/\Delta}=dd^c\varphi,\qquad\Omega_{\mathrm{H},\mathcal{X}/\Delta}=dd^c\theta. \tag{7.7}$$

Since φ and θ are subharmonic, there exists $C_0\in\mathbb{R}$ with

$$\varphi(t)\leq C_0,\qquad\theta(t)\leq C_0,\qquad t\in\Delta(1/2). \tag{7.8}$$

Since $\widetilde{\omega_{\Delta^*}}=-dd^c\log(-\log|t|)$ as a current on Δ, we deduce from (7.6) that

$$dd^c\{-C\log(-\log|t|)-\varphi\}=C\,\widetilde{\omega_{\Delta^*}}-\Omega_{\mathrm{WP},\mathcal{X}/\Delta}\geq0,$$

$$dd^c\{-C\log(-\log|t|)-\theta\}=C\,\widetilde{\omega_{\Delta^*}}-\Omega_{\mathrm{H},\mathcal{X}/\Delta}\geq0.$$

Hence $-C\log(-\log|t|)-\varphi$ and $-C\log(-\log|t|)-\theta$ are subharmonic functions on Δ, so that there exists $C_1\in\mathbb{R}$ with

$$-C\log(-\log|t|)-\varphi(t)\leq C_1,\qquad -C\log(-\log|t|)-\theta(t)\leq C_1,\qquad\forall\,t\in\Delta(1/2). \tag{7.9}$$

By (7.8) and (7.9), there exists $C_2\in\mathbb{R}$ such that for all $t\in\Delta(1/2)$,

$$-C\,\log(-\log|t|)-C_1\leq\varphi(t)\leq C_0,\qquad -C\,\log(-\log|t|)-C_1\leq\theta(t)\leq C_0. \tag{7.10}$$

Set

$$H(t):=\log\lambda(t)+A\,\theta(t)+B\,\varphi(t). \tag{7.11}$$

Since $dd^cH = 0$, $H(t)$ is a harmonic function on Δ^*. Since $\lambda(t)$ is locally L^m-integrable on Δ, the first assertion follows from (7.10) and Lemma 7.2 by setting $\chi(t) = A\,\theta(t) + B\,\varphi(t)$. The second assertion follows from (7.5), (7.7), (7.11). □

Let $g_{\mathrm{WP},\mathcal{X}/\Delta}$ be the Kähler metric on Δ^* whose Kähler form is $\omega_{\mathrm{WP},\mathcal{X}/\Delta}$.

Proposition 7.6. *Assume that $h^{1,2}(X_t) = 1$ for all $t \in \Delta^*$.*
(1) *There exists $\alpha \in \mathbb{R}$ such that as $t \to 0$:*

$$\log g_{\mathrm{WP},\mathcal{X}/\Delta}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}\right) = \alpha\, \log |t|^2 + O(\log(-\log |t|)).$$

(2) *With the same constant α as above, the following equation of currents on Δ holds:*

$$dd^c \log g_{\mathrm{WP},\mathcal{X}/\Delta}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}\right) = \alpha\,\delta_0 - \Omega_{\mathrm{H},\mathcal{X}/\Delta} + 4\,\Omega_{\mathrm{WP},\mathcal{X}/\Delta}.$$

(3) *If X_0 is a Calabi-Yau threefold with at most one ODP and if $\pi\colon \mathcal{X} \to \Delta$ is the Kuranishi family of X_0, then $\alpha = 0$.*

Proof. **(1)** Set $\lambda(t) := g_{\mathrm{WP},\mathcal{X}/\Delta}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}})$ and $A = 1$, $B = -4$ in Lemma 7.5. By the definition of Hodge form, we have $-dd^c \log \lambda = \omega_{\mathrm{H},\mathcal{X}/\Delta} - 4\,\omega_{\mathrm{WP},\mathcal{X}/\Delta}$ on Δ^*. Since $\lambda(t) \in L^1_{\mathrm{loc}}(\Delta)$ by Proposition 7.3, the result follows from Lemma 7.5 (1).
(2) The result follows from Lemma 7.5 (2).
(3) The result follows from [54, Cor. 5.1]. This completes the proof. □

If $\mathcal{X}$ is smooth, $\pi_*K_{\mathcal{X}}$ is locally free by [52, p.391, Th. V]. Since K_Δ is trivial and since $h^0(X_t, K_{\mathcal{X}}|X_t) = 1$ for $t \in \Delta^*$, $\pi_*K_{\mathcal{X}/\Delta} = \pi_*(K_{\mathcal{X}} \otimes \pi^*K_\Delta^{-1}) \cong \pi_*K_{\mathcal{X}}$ is an invertible sheaf on Δ in that case.

Lemma 7.7. *Assume that X_t is Calabi-Yau for all $t \in \Delta^*$. If $\mathcal{X}$ is smooth, there exists $\xi \in H^0(\mathcal{X}, K_{\mathcal{X}})$ such that* $\mathrm{div}(\xi) \subset X_0$.

Proof. Since $\pi_*K_{\mathcal{X}}$ is an invertible sheaf on Δ, there exists $\xi \in H^0(\mathcal{X}, K_{\mathcal{X}}) = H^0(\Delta, \pi_*K_{\mathcal{X}})$ that generates $\pi_*K_{\mathcal{X}}$ as an $\mathcal{O}_\Delta$-module, i.e., $\pi_*K_{\mathcal{X}} = \mathcal{O}_\Delta \cdot \xi$. Since $H^0(X_t, K_{\mathcal{X}}|_{X_t}) \cong H^0(X_t, K_{X_t}) \cong \mathbb{C}$ for all $t \in \Delta^*$, we get $H^0(X_t, K_{\mathcal{X}}|_{X_t}) = \mathbb{C}\,\xi|_{X_t}$ in that case by [1, Chap. 3, Th. 4.12 (ii)]. Since $K_{\mathcal{X}}|_{X_t} \cong K_{X_t} \cong \mathcal{O}_{X_t}$ for $t \in \Delta^*$, $\xi|_{X_t}$ is nowhere vanishing on X_t, $t \in \Delta^*$. This proves the lemma. □

If $\mathcal{X}$ is smooth, there exists $\xi \in H^0(\mathcal{X}, K_{\mathcal{X}})$ by Lemma 7.7 such that $\mathrm{div}(\xi) \subset X_0$. In that case, we define a section $\eta_{\mathcal{X}/\Delta} \in H^0(\mathcal{X}, K_{\mathcal{X}/\Delta})$ by $\eta_{\mathcal{X}/\Delta} := \xi \otimes (\pi^*dt)^{-1}$. We identify $\eta_{\mathcal{X}/\Delta}|_{X_t}$ with the Poincaré residue $\eta_t := \mathrm{Res}_{X_t}\xi/(\pi - t) \in H^0(X_t, K_{X_t})$ for $t \in \Delta^*$. Then

$$\xi|_{X_t} = \eta_t \otimes d\pi, \tag{7.12}$$

and $\eta_{\mathcal{X}/\Delta}$ is regarded as a family of holomorphic 3-forms. We also regard $\eta_{\mathcal{X}/\Delta}$ as the corresponding element of $H^0(\Delta, \pi_*K_{\mathcal{X}/\Delta})$.

Proposition 7.8. *Assume that $\mathcal{X}$ is smooth. Let $\eta_{\mathcal{X}/\Delta}$ be a nowhere vanishing holomorphic section of $\pi_*K_{\mathcal{X}/\Delta}$.*
(1) *There exists $\beta \in \mathbb{R}$ such that as $t \to 0$:*

$$\log \|\eta_{\mathcal{X}/\Delta}(t)\|^2_{L^2} = \beta\, \log |t|^2 + O(\log(-\log |t|)).$$

(2) *With the same constant β as above, the following equation of currents on Δ holds:*

$$dd^c \log \|\eta_{\mathcal{X}/\Delta}(t)\|^2_{L^2} = \beta\, \delta_0 - \Omega_{\mathrm{WP},\mathcal{X}/\Delta}.$$

(3) *If X_0 is a Calabi-Yau threefold with at most one ODP and if ξ is nowhere vanishing on $\mathcal{X}$, then $\log \|\eta_{\mathcal{X}/\Delta}(t)\|^2_{L^2}$ extends to a continuous function on Δ. In particular, $\beta = 0$.*

Proof. **(1)** Set $\lambda(t) := \|\eta_{\mathcal{X}/\Delta}(t)\|^2_{L^2}$ and $A = 0$, $B = 1$ in Lemma 7.5. Since

$$\begin{aligned}\int_{\Delta(1/2)} \lambda(t)\sqrt{-1}dt \wedge d\bar{t} &= \int_{\Delta(1/2)} \pi_*(\sqrt{-1}\eta_{\mathcal{X}/\Delta} \wedge \overline{\eta_{\mathcal{X}/\Delta}})\sqrt{-1}dt \wedge d\bar{t} \\ &= \int_{\pi^{-1}(\Delta(1/2))} \xi \wedge \bar{\xi} < +\infty\end{aligned}$$

by (7.12), we get $\lambda(t) \in L^1_{\mathrm{loc}}(\Delta)$. Since $-dd^c \log \lambda = \omega_{\mathrm{WP},\mathcal{X}/\Delta}$ by the definition of the Weil-Petersson form, the result follows from Lemma 7.5 (1).
(2) The result follows from Lemma 7.5 (2).
(3) The result follows from e.g. [59, Proof of Th. 8.1]. This completes the proof. $\square$

7.4. The boundary behavior of the anomaly term

In Subsection 7.4, we fix the following notation. Let $\pi\colon \mathcal{X} \to \Delta$ be a proper surjective holomorphic function on a smooth Kähler fourfold with critical locus Σ_π, so that π has relative dimension 3. Assume that $\Sigma_\pi \subset X_0$ and that X_t is a smooth Calabi-Yau threefold for all $t \in \Delta^*$.

Let $g_{\mathcal{X}}$ be a Kähler metric on $\mathcal{X}$. Let $\gamma_{\mathcal{X}}$ be the Kähler form of $g_{\mathcal{X}}$ and set $\gamma_t := \gamma_{\mathcal{X}}|_{X_t}$. Recall that the anomaly term $\mathcal{A}(X_t, \gamma_t)$ was defined in Definition 4.1. The following result is a generalization of [60, (6.17), (6.19)].

Proposition 7.9. (1) *There exists $c \in \mathbb{R}$ such that as $t \to 0$:*

$$\log \mathcal{A}(X_t, \gamma_t) = c\, \log |t|^2 + O(\log(-\log |t|)).$$

(2) *If Σ_π consists of a unique ODP and if X_0 is Calabi-Yau, then as $t \to 0$*

$$\log \mathcal{A}(X_t, \gamma_t) = -\frac{1}{12}\, \log |t|^2 + O(1).$$

Proof. **(1)** Let $g_{\mathcal{X}/\Delta}$ be the Hermitian metric on $T\mathcal{X}/\Delta$ induced from $g_{\mathcal{X}}$, and let $\gamma_{\mathcal{X}/\Delta}$ be the corresponding $(1,1)$-form on $T\mathcal{X}/\Delta$. Then we may identify $\gamma_{\mathcal{X}/\Delta}$ with the family of Kähler forms $\{\gamma_t\}_{t\in\Delta}$. Let $N^*_{X_t/\mathcal{X}}$ be the conormal bundle of X_t in $\mathcal{X}$ for $t \in \Delta^*$. Then $d\pi = \pi^*dt \in H^0(X_t, N^*_{X_t/\mathcal{X}})$ generates $N^*_{X_t/\mathcal{X}}$ for $t \in \Delta^*$, so that $N^*_{X_t/\mathcal{X}}$ is trivial in that case. Since the Hermitian metric on $\Omega^1_{X_t}$ is induced from $g_{\mathcal{X}}$ via the C^∞ identification $\Omega^1_{X_t} \cong (N^*_{X_t/\mathcal{X}})^\perp$ and since $(\gamma^3_{\mathcal{X}/\Delta}/3!)|_{X_t}$ is the volume form on $\Omega^1_{X_t}$, we get

$$\frac{\gamma^4_{\mathcal{X}}}{4!} = \frac{\gamma^3_{\mathcal{X}/\Delta}}{3!} \wedge \left(\sqrt{-1}\,\frac{d\pi}{\|d\pi\|} \wedge \frac{\overline{d\pi}}{\|d\pi\|}\right). \tag{7.13}$$

By Lemma 7.7, there exists $\xi \in H^0(\mathcal{X}, K_{\mathcal{X}})$ such that $\mathrm{div}(\xi) \subset X_0$. As before, define $\eta_{\mathcal{X}/\Delta} \in H^0(\mathcal{X}, K_{\mathcal{X}/\Delta})$ by $\eta_{\mathcal{X}/\Delta} := \xi \otimes (\pi^*dt)^{-1}$, and identify $\eta_{\mathcal{X}/\Delta}|_{X_t}$ with

the Poincaré residue $\eta_t := \mathrm{Res}_{X_t}\xi/(\pi - t) \in H^0(X_t, K_{X_t})$ for $t \in \Delta^*$. Then $\eta_{\mathcal{X}/\Delta}$ is regarded as a family of holomorphic 3-forms $\{\eta_t\}$. By (7.12) and (7.13), we get

$$(7.14)\quad \frac{\sqrt{-1}\,\eta_{\mathcal{X}/\Delta}\wedge\overline{\eta_{\mathcal{X}/\Delta}}}{\gamma^3_{\mathcal{X}/\Delta}/3!} = \frac{(-1)^3\sqrt{-1}\,\xi\wedge\overline{\xi}}{(\gamma^3_{\mathcal{X}/\Delta}/3!)\wedge d\pi\wedge\overline{d\pi}} = \frac{\xi\wedge\overline{\xi}}{\gamma^4_{\mathcal{X}/\Delta}/4!}\cdot\frac{1}{\|d\pi\|^2} = \frac{\|\xi\|^2}{\|d\pi\|^2}.$$

Let X denote a general fiber of $\pi\colon \mathcal{X}\to\Delta$. Let $\mathcal{A}(\mathcal{X}/\Delta)$ be the function on Δ^* defined by $\mathcal{A}(\mathcal{X}/\Delta)(t) := \mathcal{A}(X_t,\gamma_t)$. Then

$$(7.15)\quad \begin{aligned}\log\mathcal{A}(\mathcal{X}/\Delta) = -\frac{1}{12}\pi_*\left[\log\left(\frac{\sqrt{-1}\,\eta_{\mathcal{X}/\Delta}\wedge\overline{\eta_{\mathcal{X}/\Delta}}}{\gamma^3_{\mathcal{X}/\Delta}/3!}\right)c_3(T\mathcal{X}/\Delta, g_{\mathcal{X}/\Delta})\right]\\ +\frac{\chi(X)}{12}\log\|\eta_{\mathcal{X}/\Delta}\|^2_{L^2}.\end{aligned}$$

We use the notation in Subsection 5.3. Hence $q\colon\widetilde{\mathcal{X}}\to\mathcal{X}$ is the resolution of the Gauss maps μ and ν. Substituting (7.14) into (7.15) and using (5.1), we get (7.16)

$$\begin{aligned}\log\mathcal{A}(\mathcal{X}/\Delta) &= -\frac{1}{12}\pi_*\left[\log\left(\frac{\|\xi\|^2}{\|d\pi\|^2}\right)c_3(T\mathcal{X}/\Delta, g_{\mathcal{X}/\Delta})\right]+\frac{\chi(X)}{12}\log\|\eta_{\mathcal{X}/\Delta}\|^2_{L^2}\\ &= -\frac{1}{12}\widetilde{\pi}_*\left[\log q^*\left(\frac{\|\xi\|^2}{\|d\pi\|^2}\right)\widetilde{\mu}^*c_3(U,g_U)\right]+\frac{\chi(X)}{12}\log\|\eta_{\mathcal{X}/\Delta}\|^2_{L^2}.\end{aligned}$$

Since $\mathrm{div}(q^*\xi)\subset\widetilde{\pi}^{-1}(0)$ by the condition $\mathrm{div}(\xi)\subset X_0$, the assertion follows from Lemma 5.8 and Proposition 7.8 (1) applied to the second line of (7.16).
(2) Assume that Σ_π consists of a unique ODP and that X_0 is Calabi-Yau. We use the notation in Subsect. 5.9. We may assume by Lemma 6.1 that ξ is nowhere vanishing on $\mathcal{X}$. Hence $\mathrm{div}(q^*\xi)=\emptyset$, and $\widetilde{\pi}_*\{q^*\log\|\xi\|^2\,\widetilde{\mu}^*c_3(U,g_U)\}$ and $\log\|\eta_{\mathcal{X}/\Delta}\|^2_{L^2}$ are bounded as $t\to 0$ by the first equation of Lemma 5.8 and by Proposition 7.8 (3). We deduce from (7.16) that

$$(7.17)\quad \log\mathcal{A}(\mathcal{X}/\Delta) = \frac{1}{12}\widetilde{\pi}_*\{q^*(\log\|d\pi\|^2)\,\widetilde{\mu}^*c_3(U,g_U)\}+O(1).$$

Since $E=\mathbb{P}^3$ and $\varphi=(-1)^3\widetilde{\mu}^*c_3(U)$ in the second equation of Lemma 5.8, we get

$$(7.18)\quad \log\mathcal{A}(\mathcal{X}/\Delta)(t) = \left(\frac{1}{12}\int_{\mathbb{P}^3}c_3(U)\right)\log|t|^2+O(1) = \frac{(-1)^3}{12}\log|t|^2+O(1).$$

This proves (2). □

7.5. The Weil-Petersson and Hodge metrics on the Kuranishi space

In Subsect. 7.5, we fix the following notation. Let X be a smoothable Calabi-Yau threefold with only one ODP as its singular set, and let $\mathfrak{p}\colon(\mathfrak{X},X)\to(\mathrm{Def}(X),[X])$ be the Kuranishi family with discriminant locus $\mathfrak{D}$. Assume that $\dim\mathrm{Def}(X) = h^{1,2}(X)=1$.

By Lemma 6.1, there exists a nowhere vanishing holomorphic 4-form ξ on $\mathfrak{X}$. Then $\eta_{\mathfrak{X}/\mathrm{Def}(X)} = \xi\otimes\pi^*(ds)^{-1}$ is a nowhere vanishing holomorphic section of $\mathfrak{p}_*K_{\mathfrak{X}/\mathrm{Def}(X)}$. Set $\eta_s := \eta_{\mathfrak{X}/\mathrm{Def}(X)}|_{X_s}$. We identify η_s with the corresponding holomorphic 3-form on $(X_s)_{\mathrm{reg}}$ such that $\eta_s\otimes(ds)=\xi|_{X_s}$ under the canonical isomorphism $K_{X_s}\otimes\mathfrak{p}^*K_{\mathrm{Def}(X)}|_{X_s} = K_{\mathfrak{X}}|_{X_s}$. Then $\{\eta_s\}_{s\in S}$ is regarded as a holomorphic family of nowhere vanishing holomorphic 3-forms.

For $p = 0, 1$ and $q \geq 0$, the direct image sheaves $R^q \mathfrak{p}_* \Omega^p_{\mathfrak{X}/\mathrm{Def}(X)}$ are locally free by Definition 2.1 (ii) and Theorem 2.11. For $p = 0, 1$, let σ_p be a nowhere vanishing holomorphic section of $\lambda(\Omega^p_{\mathfrak{X}/\mathrm{Def}(X)})$.

By Proposition 2.8, there exists a Kähler metric $g_{\mathfrak{X}}$ on $\mathfrak{X}$. Let $g_{\mathfrak{X}/\mathrm{Def}(X)}$ be the Hermitian metric on $T\mathfrak{X}/\mathrm{Def}(X)|_{\mathfrak{X}\setminus\Sigma_{\mathfrak{p}}}$ induced from $g_{\mathfrak{X}}$. Set $g_s := g_{\mathfrak{X}}|_{X_s}$ for $s \in \mathrm{Def}(X)$.

Theorem 7.10. *The following formula holds for $p = 0, 1$:*

$$\log \|\sigma_p(s)\|^2_{\lambda(\Omega^p_{\mathfrak{X}/\mathrm{Def}(X)}), L^2, g_{\mathfrak{X}/\mathrm{Def}(X)}} = O(\log(-\log|s|)).$$

Proof. Let $p = 0$. Let 1 be the section of $\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}}$ such that $1_s = 1 \in H^0(X_s, \mathcal{O}_{X_s})$. Regard $\eta_{\mathfrak{X}/\mathrm{Def}(X)}$ as a nowhere vanishing holomorphic section of $(R^3\mathfrak{p}_*\mathcal{O}_{\mathfrak{X}})^\vee$ by the relative Serre duality. Set $\sigma_0 := 1 \otimes \eta_{\mathfrak{X}/\mathrm{Def}(X)}$. Since

$$\log \|\sigma_0(s)\|^2_{L^2, g_s} = \log \mathrm{Vol}(X_s, g_s) + \log \|\eta_s\|^2_{L^2} = \log \|\eta_s\|^2_{L^2} + O(1),$$

the assertion for $p = 0$ follows from Proposition 7.8 (3).

Let $p = 1$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{b_2(X)}$ be a $\mathbb{Z}$-basis of $H^2(X, \mathbb{Z})/\mathrm{Torsion}$. There exist holomorphic line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_{b_2(X)}$ on $\mathfrak{X}$ by Lemma 2.16 such that $c_1(\mathcal{L}_i)|_X = \mathbf{e}_i$ for $1 \leq i \leq b_2(X)$, and such that the Dolbeault cohomology classes of their Chern forms $\mathfrak{C}_1(\mathcal{L}_1), \ldots, \mathfrak{C}_1(\mathcal{L}_{b_2(X)})$ form a local basis of $R^1\pi_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ as a $\mathcal{O}_{\mathrm{Def}(X)}$-module.

By Theorem 6.2, $(\rho_s^\vee)^{-1}(ds) \otimes \eta_s^{-1}$ is a local basis of $R^2\pi_*\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)}$ as an $\mathcal{O}_{\mathrm{Def}(X)}$-module. For $s \in \mathrm{Def}(X)$, set

$$\sigma_1(s) := (\mathfrak{C}_1(\mathcal{L}_1) \wedge \cdots \wedge \mathfrak{C}_1(\mathcal{L}_{b_2(X)}))^{-1} \otimes ((\rho_s^\vee)^{-1}(ds) \otimes \eta_s^{-1}).$$

Then σ_1 is a nowhere vanishing holomorphic section of $\lambda(\Omega^1_{\mathfrak{X}/\mathrm{Def}(X)})$.

Let γ_s be the Kähler form of $g_{\mathfrak{X}}|_{X_s}$. Since $g_{\mathfrak{X}}$ is a Kähler metric on $\mathfrak{X}$, the section $\mathrm{Def}(X) \ni s \to [\gamma_s] \in H^2(X_s, \mathbb{R})$ of $R^2\mathfrak{p}_*\mathbb{R}$ is constant. Let $[\gamma] \in H^2(X, \mathbb{R})$ be the element corresponding to $[\gamma_s]$. By Lemma 4.12,

$$\|\mathfrak{C}_1(\mathcal{L}_1) \wedge \cdots \wedge \mathfrak{C}_1(\mathcal{L}_{b_2(X)})\|^2_{L^2, g_s}(s) = \mathrm{Vol}_{L^2}(H^2(X, \mathbb{Z}), [\gamma]) \neq 0$$

is a constant function on $\mathrm{Def}(X)$. Hence we get

$$\begin{aligned} \log \|\sigma_1(s)\|^2_{L^2, g_s} &= -\log \mathrm{Vol}_{L^2}(H^2(X, \mathbb{Z}), [\gamma]) - \log g_{\mathrm{WP}}(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) - h^{1,2}(X) \log \|\eta_s\|^2_{L^2} \\ &= O(\log(-\log|s|)) \end{aligned}$$

by Propositions 4.4, 7.6 (3) and 7.8 (3). This proves the theorem. $\square$

8. The singularity of the BCOV invariant I – the case of ODP

In Sect. 8, we fix the following notation. Let $\pi \colon \mathcal{X} \to S$ be a proper, surjective, flat holomorphic map from a compact, connected smooth Kähler fourfold to a compact Riemann surface. Let $\mathcal{D}$ be the discriminant locus and let $0 \in \mathcal{D}$. We assume that $X := X_0$ is a Calabi-Yau threefold with a unique ODP as its singular set satisfying $h^2(\Omega^1_X) = 1$. The deformation germ $\pi \colon (\mathcal{X}, X) \to (S, 0)$ is a smoothing of X, and a general fiber of π is a smooth Calabi-Yau threefold. We set $o := \mathrm{Sing}\, X$.

Let $\mathfrak{p} \colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X with discriminant locus $\mathfrak{D} = [X]$. Since $h^2(\Omega^1_X) = 1$, we have $\dim \mathrm{Def}(X) = 1$. By Proposition 2.8, $\mathfrak{X}$ is Kähler. Let $g_{\mathfrak{X}}$ be a Kähler metric on $\mathfrak{X}$, and set $g_{\mathfrak{X}/\mathrm{Def}(X)} := g_{\mathfrak{X}}|_{T\mathfrak{X}/\mathrm{Def}(X)}$.

Let $\mu\colon (S,0)\to(\mathrm{Def}(X),[X])$ be the holomorphic map that induces the family $\pi\colon(\mathcal{X},X)\to(S,0)$ from the Kuranishi family. By the local description (2.2), we have $\mathcal{O}_{\mathcal{X},o}\cong\mathbb{C}\{z_0,z_1,z_2,z_3\}/(z_0^2+\cdots+z_3^2-\mu(t))$. Since $\mathcal{X}$ is smooth, $\mathfrak{D}=\mu(0)$ is not a critical value of μ, and the morphism of germs $\mu\colon(S,0)\to(\mathrm{Def}(X),[X])$ is an isomorphism. Hence there exist a neighborhood $\mathcal{U}$ of $0\in S$ and an isomorphism of families $f\colon\mathcal{X}|_{\mathcal{U}}\cong\mathfrak{X}|_{\mu(\mathcal{U})}$.

Let $g_{\pi^{-1}(\mathcal{U})}$ be the Kähler metric on $\pi^{-1}(\mathcal{U})$ defined as

$$g_{\pi^{-1}(\mathcal{U})}=f^*g_{\mathfrak{X}}.$$

Let $g_{\mathcal{X}/S}$ be the Hermitian metric on $T\mathcal{X}/S|_{\pi^{-1}(\mathcal{U})\setminus\Sigma_\pi}$ induced from $g_{\pi^{-1}(\mathcal{U})}$. Then

$$g_{\mathcal{X}/S}=f^*g_{\mathfrak{X}/\mathrm{Def}(X)}.$$

Let $\|\cdot\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}$ be the L^2-metric on the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}$ with respect to $g_{\mathcal{X}/S}$. Since $\mathcal{F}^p_{\mathcal{X}/S}$ is acyclic on $\mathcal{X}$ for $p=0,1$, we have the following isomorphisms for $p=0,1$:

$$\lambda(\mathcal{E}^p_{\mathcal{X}/S})|_{\mathcal{U}}\cong\mu^*\lambda(\Omega^p_{\mathfrak{X}/\mathrm{Def}(X)}),\qquad \|\cdot\|_{L^2,g_{\mathcal{X}/S}}=\mu^*\|\cdot\|_{L^2,g_{\mathfrak{X}/\mathrm{Def}(X)}}.\tag{8.1}$$

Let t be a local coordinate of S centered at 0. Let σ_p be a nowhere vanishing holomorphic section of the Kähler extension $\lambda(\mathcal{E}^p_{\mathcal{X}/S})$ near $0\in\mathcal{D}$.

Theorem 8.1. *The following formula holds as $t\to0$:*

$$(-1)^p\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}=\begin{cases}O(\log(-\log|t|)) & (p=0,1)\\ -\log|t|^2+O(\log(-\log|t|)) & (p=2,3).\end{cases}$$

Proof. Let $p=0,1$. Since $\mu\colon(S,0)\to(\mathrm{Def}(X),[X])$ is an isomorphism, the assertion follows from Theorem 7.10 and (8.1).

Let $p=2,3$. Recall that the canonical element $\mathbf{1}_{p,3-p}(X_t)\in\lambda(\Omega^p_{X_t})\otimes\lambda(\Omega^{3-p}_{X_t})^\vee$ was defined in Subsection 3.3. Let $\mathbf{1}_{p,3-p,S^o}$ be the nowhere vanishing holomorphic section of $\lambda(\Omega^p_{\mathcal{X}^o/S^o})\otimes\lambda(\Omega^{3-p}_{\mathcal{X}^o/S^o})^\vee$ defined by

$$\mathbf{1}_{p,3-p,S^o}(t):=\mathbf{1}_{p,3-p}(X_t)\in\lambda(\Omega^p_{X_t})\otimes\lambda(\Omega^{3-p}_{X_t})^\vee,\qquad t\in S^o.$$

Then

$$\|\mathbf{1}_{p,3-p,S^o}(t)\|_{L^2,g_{\mathcal{X}/S}}=\|\mathbf{1}_{p,3-p,S^o}(t)\|_{Q,g_{\mathcal{X}/S}}=1,\qquad t\in S^o.\tag{8.2}$$

by Proposition 3.4.

By Theorem 5.11, we get

$$\begin{aligned}&\log\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,Q,g_{\mathcal{X}/S}}\\&=(-1)^{3-p}\delta(3,p)\log|t|^2+(-1)^3\cdot(-1)^{3-(3-p)}\delta(3,3-p)\log|t|^2+O(1)\\&=(-1)^{3-p}\log|t|^2+O(1),\end{aligned}\tag{8.3}$$

where we used the first identity of Lemma 5.12 to get the last equality of (8.3).

Set

$$f_p(t):=\frac{\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}}{\mathbf{1}_{p,3-p}(t)}\in\mathcal{O}(S^o).$$

By (8.2), we get

$$
\begin{aligned}
&\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,Q,g_{\mathcal{X}/S}}\\
&=|f_p(t)|^2\cdot\|\mathbf{1}_{p,3-p}(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,Q,g_{\mathcal{X}/S}}\\
&=|f_p(t)|^2\cdot\|\mathbf{1}_{p,3-p}(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,L^2,g_{\mathcal{X}/S}}\\
&=\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,L^2,g_{\mathcal{X}/S}},
\end{aligned}
\tag{8.4}
$$

which, together with (8.3), yields that

$$
\log\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,L^2,g_{\mathcal{X}/S}}=(-1)^{3-p}\,\log|t|^2+O(1).
\tag{8.5}
$$

By Theorem 8.1 for $p=0,1$ and (8.4), we get

$$
\begin{aligned}
&(-1)^p\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}\\
&=(-1)^p\,\log\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,L^2,g_{\mathcal{X}/S}}\\
&\quad+(-1)^p\log\|\sigma_{3-p}(t)\|^2_{\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}\\
&=-\log|t|^2+O(\log(-\log|t|)).
\end{aligned}
$$

This proves the theorem for $p=2,3$. □

Let γ_t be the Kähler form of $g_{\mathcal{X}/S}|_{X_t}$.

Theorem 8.2. *The following formula holds as $t\to 0$:*

$$
\log\tau_{\mathrm{BCOV}}(X_t)=\frac{1}{6}\log|t|^2+O(\log(-\log|t|)).
$$

Proof. By the definition of the BCOV torsion of (X_t,γ_t), we have

$$
\begin{aligned}
\log\mathcal{T}_{\mathrm{BCOV}}(X_t,\gamma_t)&=\sum_{p\geq 0}(-1)^p p\,\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}\\
&\quad-\sum_{p\geq 0}(-1)^p p\,\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}.\\
&=-\frac{19}{4}\log|t|^2+\sum_{p=2}^{3}p\log|t|^2+O(\log(-\log|t|^2))\\
&=\frac{1}{4}\log|t|^2+O(\log(-\log|t|^2)),
\end{aligned}
\tag{8.6}
$$

where we used Theorems 5.13 and 8.1 to get the second equality. Since

$$
\log\mathrm{Vol}(X_t,\gamma_t)=O(1),\qquad \log\mathrm{Vol}_{L^2}(H^2(X_t,\mathbb{Z}),[\gamma_t])=O(1),
$$

we deduce from Proposition 7.9 (2) and (8.6) that

$$
\begin{aligned}
\log\tau_{\mathrm{BCOV}}(X_t)&=\log\mathcal{A}(X_t,\gamma_t)+\log\mathcal{T}_{\mathrm{BCOV}}(X_t,\gamma_t)+O(1)\\
&=\frac{1}{6}\log|t|^2+O(\log(-\log|t|^2)).
\end{aligned}
$$

This proves the theorem. □

9. The singularity of the BCOV invariant II – general degenerations

In Section 9, we fix the following notation: Let $\mathcal{X}$ be an irreducible projective algebraic fourfold and let S be a compact Riemann surface. Let $\pi\colon \mathcal{X} \to S$ be a surjective, flat holomorphic map. Let $\mathcal{D} \subset S$ be a reduced divisor and set $\mathcal{X}^o := \mathcal{X} \setminus \pi^{-1}(\mathcal{D})$, $S^o := S \setminus \mathcal{D}$, $\pi^o := \pi|_{\mathcal{X}^o}$. Let $0 \in \mathcal{D}$, and let (U, t) be a coordinate neighborhood of S centered at 0 such that $U \setminus \{0\} \cong \varDelta^*$.

In Section 9, we shall prove a generalization of Theorem 8.2.

Theorem 9.1. *If $\pi^o\colon \mathcal{X}^o \to S^o$ is a smooth morphism whose fibers are Calabi-Yau threefolds, then there exists $\alpha \in \mathbb{R}$ such that as $t \to 0$,*

$$\log \tau_{\rm BCOV}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|^2)).$$

First, we shall prove Theorem 9.1 when $\pi\colon \mathcal{X} \to S$ is a semi-stable family. Then we shall reduce the general case to this particular case by the semi-stable reduction theorem of Mumford [27]. We set $D := X_0$ in this section.

9.1. The singularity of L^2 metrics for semi-stable degenerations

In Subsections 9.1 and 9.2, we assume that $\mathcal{X}$ is smooth and that $D = X_0$ is a reduced divisor of normal crossing, i.e., for every $x \in D$, there exist integers $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$ and a coordinate neighborhood $(\mathcal{U}, (z_0, z_1, z_2, z_3))$ of $\mathcal{X}$ centered at x such that

$$\pi(z) = z_0^{\epsilon_0} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3}, \qquad z \in \mathcal{U}.$$

Let $\Omega^1_{\mathcal{X}/S}(\log D)$ be the sheaf of meromorphic 1-forms on $\mathcal{X}$ with logarithmic pole along D. Then $\Omega^1_{\mathcal{X}}(\log D)|_{\mathcal{X}\setminus D} = \Omega^1_{\mathcal{X}}|_{\mathcal{X}\setminus D}$, and $\Omega^1_{\mathcal{X}}(\log D)|_{\mathcal{U}}$ is a free $\mathcal{O}_{\mathcal{U}}$-module generated by $dz_0/z_0^{\epsilon_0}$, $dz_1/z_1^{\epsilon_1}$, $dz_2/z_2^{\epsilon_2}$, $dz_3/z_3^{\epsilon_3}$.

Let $\Omega^1_S(\log 0)$ be the sheaf of meromorphic 1-forms on S with logarithmic pole at 0. Then $\Omega^1_S(\log 0)_0 = \mathcal{O}_{S,0}\, dt/t$. We set

$$\Omega^1_{\mathcal{X}/S}(\log D) := \Omega^1_{\mathcal{X}}(\log D)/\pi^*\Omega^1_S(\log 0).$$

See e.g. [51, Sect. 2], [56, Chap. 3, Sect. 2] for more details about $\Omega^1_{\mathcal{X}/S}(\log D)$.

Let $g_{\mathcal{X}}$ be a Kähler metric on $\mathcal{X}$ whose Kähler class is integral. Let $\kappa \in H^2(\mathcal{X}, \mathbb{Z})$ be the Kähler class of $g_{\mathcal{X}}$. We set $g_{\mathcal{X}/S} := g_{\mathcal{X}}|_{T\mathcal{X}/S}$.

9.1.1. *The canonical extension of the Hodge bundles.* For the proof of Theorem 9.1, let us recall some results of Schmid [48] and Steenbrink [51]. Set $U^o := U \setminus \{0\}$. We fix $b \in U^o$ and set $W := H^m(X_b, \mathbb{C})$ and $l := \dim W$.

Let ${}^o\mathbf{H}^m := R^m\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{U^o}$ and consider the Gauss-Manin connection on ${}^o\mathbf{H}^m$. The canonical extension $\mathbf{H}^m$ of ${}^o\mathbf{H}^m$ from U^o to U is defined as follows: Let $\{v_1, \ldots, v_l\}$ be a basis of W, and let $\gamma \in GL(W)$ be the Picard-Lefschetz transformation. There exists a Nilpotent operator $N \in \mathrm{End}(W)$ with $\gamma = \exp(N)$.

Let $\psi\colon \widetilde{U^o} \ni z \to \exp(2\pi\sqrt{-1}z) \in U^o$ be the universal covering. Since ${}^o\mathbf{H}^m$ is flat, the vectors v_i extend to flat holomorphic sections $\mathbf{v}_i \in \Gamma(\widetilde{U^o}, \psi^*({}^o\mathbf{H}^m))$, which induce an isomorphism $\psi^*({}^o\mathbf{H}^m) \cong \mathcal{O}_{\widetilde{U^o}} \otimes_{\mathbb{C}} W$ of flat bundles. Under this trivialization of $\psi^*({}^o\mathbf{H}^m)$, we have $\mathbf{v}_i(z+1) = \gamma \cdot \mathbf{v}_i(z)$ for all i. After Schmid [48, pp.234-236], we define holomorphic frame fields of $\psi^*({}^o\mathbf{H}^m)$ by

$$\mathbf{s}_i(\exp 2\pi\sqrt{-1}z) := \exp(-z\,N)\ \mathbf{v}_i(z) = \sum_{k\geq 0} \frac{1}{k!}(-z\,N)^k \mathbf{v}_i(z). \tag{9.1}$$

Since $\mathbf{s}_1, \ldots, \mathbf{s}_l \in \Gamma(\widetilde{U^o}, \psi^*({}^o\mathbf{H}^m))$ are invariant under the translation $z \to z+1$, they descend to single-valued holomorphic frame fields of ${}^o\mathbf{H}^m$. Then $\mathbf{H}^m$ is a locally free sheaf on U defined as $\mathbf{H}^m := \mathcal{O}_U\, \mathbf{s}_1 \oplus \cdots \oplus \mathcal{O}_U\, \mathbf{s}_l$.

By Hodge theory, ${}^o\mathbf{H}^m$ carries the *Hodge filtration* $0 \subset {}^o\mathbf{F}^m \subset \cdots \subset {}^o\mathbf{F}^1 \subset {}^o\mathbf{H}^m$ such that ${}^o\mathbf{F}^p$ is a holomorphic subbundle of ${}^o\mathbf{H}^m$ with ${}^o\mathbf{F}^p/{}^o\mathbf{F}^{p+1} \cong R^{m-p}\pi_*\Omega^p_{\mathcal{X}/S}|_{U^o}$. For $t \in U^o$, we have the natural identification ${}^o\mathbf{F}^p_t = \bigoplus_{i \geq p} H^{m-i}(X_t, \Omega^i_{X_t})$.

By [48, p.235], [51, Th. 2.11], [63, pp.130 Cor.], the filtration $\{{}^o\mathbf{F}^p\}$ extends to a filtration $\{\mathbf{F}^p\}$ of $\mathbf{H}^m$ such that $\mathbf{F}^p/\mathbf{F}^{p+1} \cong R^{m-p}\pi_*\Omega^p_{\mathcal{X}/S}(\log D)|_U$. Under this isomorphism, we have an identification of holomorphic line bundles on U:

$$i_p : (\det \mathbf{F}^p) \otimes (\det \mathbf{F}^{p+1})^{-1} \cong \det R^{m-p}\pi_*\Omega^p_{\mathcal{X}/S}(\log D)|_U. \tag{9.2}$$

Since ${}^o\mathbf{H}^m_t = H^m(X_t, \mathbb{C})$ for $t \in U^o$, ${}^o\mathbf{H}^m$ is equipped with the L^2-metric $h_{R^m\pi_*\mathbb{C}}$ with respect to $g_{\mathcal{X}/S}$. Recall that the C^∞ vector bundles $\mathcal{K}^{p,q}(\mathcal{X}^o/U^o)$ on U^o were defined in Subsect. 3.5. Let $h_{\mathbf{F}^p}$ be the L^2-metric on ${}^o\mathbf{F}^p$ induced from $h_{R^m\pi_*\mathbb{C}}$ by the C^∞ isomorphism ${}^o\mathbf{F}^p \cong \bigoplus_{i \geq p} \mathcal{K}^{i,m-i}(\mathcal{X}^o/U^o)$. By the definition of L^2-metrics, the isomorphism $i_p|_{U^o}$ induces an isometry of Hermitian line bundles on U^o:

$$\big((\det {}^o\mathbf{F}^p) \otimes (\det {}^o\mathbf{F}^{p+1})^{-1}, \det h_{\mathbf{F}^p} \otimes (\det h_{\mathbf{F}^{p+1}})^{-1}\big) \cong (\det R^{m-p}\Omega^p_{\mathcal{X}/S}, \|\cdot\|_{L^2}). \tag{9.3}$$

Recall that the Kähler operator $L\colon H^m(X_t, \mathbb{C}) \to H^{m+2}(X_t, \mathbb{C})$ with respect to $\kappa|_{X_t}$ was defined in Subsect. 4.4.1. Then L induces a homomorphism of $\mathcal{O}_U$-modules $L\colon \mathbf{H}^m \to \mathbf{H}^{m+2}$. The primitive part of $\mathbf{H}^m$ is the holomorphic flat subbundle of $\mathbf{H}^m$ defined as $\mathbf{P}^m := \mathbf{H}^m \cap \ker L^{4-m}$. The Picard-Lefschetz transformation γ preserves $\mathbf{P}^m$. If $\mathbf{s}_i \in \Gamma(U, \mathbf{P}^m)$, there exists $k \in \mathbb{Z}$, $C \in \mathbb{R}$ by [48, p.252 Th. 6.6'] such that

$$\|\mathbf{s}_i(t)\|^2_{L^2} \leq C\,(-\log|t|)^k, \qquad t \in U^o. \tag{9.4}$$

9.1.2. *Singularities of the L^2-metrics: the case of canonical extension.*

Lemma 9.2. *Let $m = 3$. Let $\mathbf{f}_p$ be a nowhere vanishing holomorphic section of* $\det \mathbf{F}^p$ *defined on U. Then there exists $c_p \in \mathbb{R}$ such that as $t \to 0$,*

$$\log\|\mathbf{f}_p(t)\|^2_{L^2} = c_p\,\log|t|^2 + O(\log(-\log|t|)).$$

Proof. Since $m = 3$, we have $\mathbf{H}^3 = \mathbf{P}^3$, i.e., the groups $H^3(X_t, \mathbb{C})$ are primitive. By (9.4), there exists a constant $C > 0$ and $l \in \mathbb{Z}$ such that

$$\lambda_p(t) := \|\mathbf{f}_p(t)\|^2_{L^2} \leq C\,(-\log|t|)^l, \qquad t \in U^o. \tag{9.5}$$

We set $\lambda_4(t) = 1$. By Proposition 4.6 and (9.3), we get the following on U^o:

$$-dd^c(\log\lambda_p - \log\lambda_{p+1}) = \begin{cases} -\omega_{\mathrm{WP},\mathcal{X}^o/U^o} & (p=0) \\ -\omega_{\mathrm{H},\mathcal{X}^o/U^o} + 3\,\omega_{\mathrm{WP},\mathcal{X}^o/U^o} & (p=1) \\ \omega_{\mathrm{H},\mathcal{X}^o/U^o} - 3\,\omega_{\mathrm{WP},\mathcal{X}^o/U^o} & (p=2) \\ \omega_{\mathrm{WP},\mathcal{X}^o/U^o} & (p=3). \end{cases} \tag{9.6}$$

Since $\lambda_p \in L^1_{\mathrm{loc}}(U)$ by (9.5), the result follows from Lemma 7.5 (1) and (9.6). $\square$

Let σ_p be a nowhere vanishing holomorphic section of $\lambda(\mathcal{E}^p_{\mathcal{X}/S})$ near 0.

Proposition 9.3. *There exists $\beta_0 \in \mathbb{R}$ such that as $t \to 0$:*

$$\log\|\sigma_0(t)\|^2_{\lambda(\mathcal{O}_{\mathcal{X}}), L^2, g_{\mathcal{X}/S}} = \beta_0\,\log|t|^2 + O(\log(-\log|t|)).$$

Proof. We may assume that $\sigma_0 = \mathbf{f}_0 \otimes \mathbf{f}_1^{-1}$ under the isomorphism (9.2). Since (9.2) induces the isometry (9.3), the result follows from Lemma 9.2. □

By [51, Th. 2.11], $R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D)$ is locally free. Set $r := \operatorname{rk} R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D)$. Let $e_1(t), \ldots, e_r(t)$ be a basis of $R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D)$ as a free $\mathcal{O}_U$-module.

Proposition 9.4. *For $0 \leq q \leq 3$, there exists $\delta_q \in \mathbb{R}$ such that as $t \to 0$,*

$$\log \|e_1(t) \wedge \cdots \wedge e_r(t)\|^2_{\det R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D), L^2, g_{\mathcal{X}/S}} = \delta_q \log |t|^2 + O(\log(-\log |t|)).$$

Proof. Since $r = 0$ when $q = 0, 3$, it suffices to prove the cases $q = 1, 2$.

(Case 1) Let $q = 2$. There exists a nowhere vanishing holomorphic function $h(t)$ on U such that $e_1(t) \wedge \cdots \wedge e_r(t) = h(t)\,\mathbf{f}_1(t) \otimes \mathbf{f}_2(t)^{-1}$ under the isomorphism (9.2). Since (9.2) induces the isometry (9.3), the result follows from Lemma 9.2.

(Case 2) Let $q = 1$. When $m = 2$, we have $\mathbf{H}^2 = \mathbf{F}^1$. Hence $r = l$. Identify the integral Kähler class κ on $\mathcal{X}$ with the corresponding flat section of $\mathbf{H}^2$. Then $\mathbf{P}^m$ and $\mathcal{O}_U\,\kappa$ are holomorphic flat subbundles of $\mathbf{H}^m$ preserved by the Picard-Lefschetz transformation γ. Hence we have a decomposition $\mathbf{H}^2 = \mathbf{P}^2 \oplus \mathcal{O}_U\,\kappa$ of γ-invariant flat bundles on U. Choose $v_1 = \kappa_b$ and $v_2, \ldots, v_l \in \mathbf{P}^2_b \cap H^2(X_b, \mathbb{Z})/\text{Torsion}$ in Subsect. 9.1.1. Then $\mathbf{s}_1 = \kappa$ and $\mathbf{P}^m = \mathcal{O}_U\,\mathbf{s}_2 \oplus \cdots \oplus \mathcal{O}_U\,\mathbf{s}_l$. Since $\mathbf{v}_1(z), \ldots, \mathbf{v}_l(z)$ are identified with $v_1, \ldots, v_l$ via the C^∞ trivialization $\mathcal{X}^o \times_{U^o} \widetilde{U}^o \cong X_b \times \widetilde{U}^o$, we get by Definition 4.11 and Lemma 4.12

$$\|\mathbf{v}_1(z) \wedge \cdots \wedge \mathbf{v}_l(z)\|^2_{L^2,\kappa} = \mathrm{Vol}_{L^2}(H^2(X_b, \mathbb{Z}), \kappa_b), \qquad \forall\, z \in \widetilde{U}^o. \tag{9.7}$$

Since N is nilpotent and hence $\det \exp(-zN) = 1$ for all $z \in \widetilde{U}^o$, we get

$$\begin{aligned}
\mathbf{s}_1(e^{2\pi\sqrt{-1}z}) \wedge \cdots \wedge \mathbf{s}_l(e^{2\pi\sqrt{-1}z}) &= \exp(-z\,N)\,\mathbf{v}_1(z) \wedge \cdots \wedge \exp(-z\,N)\,\mathbf{v}_l(z) \\
&= \det \exp(-zN) \cdot \mathbf{v}_1(z) \wedge \cdots \wedge \mathbf{v}_l(z) \\
&= \mathbf{v}_1(z) \wedge \cdots \wedge \mathbf{v}_l(z).
\end{aligned} \tag{9.8}$$

By (9.7), (9.8), we get for all $t \in U^o$:

$$\|\mathbf{s}_1(t) \wedge \cdots \wedge \mathbf{s}_l(t)\|^2_{L^2,\kappa} = \mathrm{Vol}_{L^2}(H^2(X_b, \mathbb{Z}), \kappa_b). \tag{9.9}$$

Since $\{\mathbf{s}_1(t), \ldots, \mathbf{s}_l(t)\}$ is a basis of $R^1\pi_*\Omega^1_{\mathcal{X}/S}(\log D)$ as a free $\mathcal{O}_S$-module, the result follows from (9.9). This completes the proof. □

9.1.3. *Comparison of the Kähler extension and the canonical extension.*

Proposition 9.5. *There exists $\beta_1 \in \mathbb{R}$ such that*

$$\log \|\sigma_1(t)\|^2_{\lambda(\Omega^1_{\mathcal{X}/S}), L^2, g_{\mathcal{X}/S}} = \beta_1 \log |t|^2 + O(\log(-\log |t|)) \qquad (t \to 0).$$

Proof. Consider the natural injection $0 \to \Omega^1_{\mathcal{X}/S} \to \Omega^1_{\mathcal{X}/S}(\log D)$, and set $Q := \Omega^1_{\mathcal{X}/S}(\log D)/\Omega^1_{\mathcal{X}/S}$. Then Q is a torsion sheaf on $\mathcal{X}$ whose support is contained in $\mathrm{Sing}(D)$. Consider the long exact sequence of direct image sheaves induced by the short exact sequence of sheaves $0 \to \Omega^1_{\mathcal{X}/S} \to \Omega^1_{\mathcal{X}/S}(\log D) \to Q \to 0$ on $\mathcal{X}$:

$$R^{q-1}\pi_*\Omega^1_{\mathcal{X}/S}(\log D) \to R^{q-1}\pi_*Q \to R^q\pi_*\Omega^1_{\mathcal{X}/S} \to R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D) \to R^q\pi_*Q.$$

Since $R^q\pi_*Q$ is a torsion sheaf on U supported at $\{0\}$ for all q, there exist torsion sheaves M_q, N_q on U supported at $\{0\}$ and an exact sequence of coherent sheaves on U:

$$0 \to M_q \to R^q\pi_*\Omega^1_{\mathcal{X}/S} \xrightarrow{\quad j \quad} R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D) \to N_q \to 0. \tag{9.10}$$

Since $U \cong \varDelta$ and hence $\mathcal{O}_{U,t}$ is a discrete valuation ring for all $t \in U$, the image $j(R^q\pi_*\Omega^1_{\mathcal{X}/S})$ is a locally free submodule of $R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D)$. Hence $(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor}$, the torsion part of $R^q\pi_*\Omega^1_{\mathcal{X}/S}$, is contained in $\ker j$. Since $M_q \subset (R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor}$, we have

$$M_q = (R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor}. \tag{9.11}$$

Since $N_q = R^q\pi_*\Omega^1_{\mathcal{X}/S}(\log D)/j(R^q\pi_*\Omega^1_{\mathcal{X}/S})$ is a torsion sheaf, there exist integers $\nu_1,\dots,\nu_r \geq 0$ such that $N_q \cong \mathbb{C}\{t\}/(t^{\nu_1})\oplus\cdots\oplus\mathbb{C}\{t\}/(t^{\nu_r})$ and $j(R^q\pi_*\Omega^1_{\mathcal{X}/S}) = \mathcal{O}_U\,t^{\nu_1}e_1(t)\oplus\cdots\oplus\mathcal{O}_U\,t^{\nu_r}e_r(t)$. Hence

$$\det j(R^q\pi_*\Omega^1_{\mathcal{X}/S}) = \mathcal{O}_U\cdot t^{\nu_1}e_1(t)\wedge\cdots\wedge t^{\nu_r}e_r(t). \tag{9.12}$$

By [1, p.110 3. Proof of the theorem], there exists a complex of locally free sheaves of finite rank on U

$$E_\bullet\colon 0 \to E_{-1} \xrightarrow{v_{-1}} E_0 \xrightarrow{v_0} \cdots \xrightarrow{v_{k-1}} E_k \to 0$$

such that $R^q\pi_*\Omega^1_{\mathcal{X}/S}$ is the q-th cohomology sheaf of $E_\bullet$, i.e., $R^q\pi_*\Omega^1_{\mathcal{X}/S} \cong H^q(E_\bullet)$ for all $q \geq 0$. Since $U \cong \varDelta$, $\ker v_q \subset E_q$ and $\operatorname{Im} v_q \subset E_{q+1}$ are locally free sheaves on U for all $q \geq -1$. Let ξ_q be the inverse image of $(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor}$ by the natural surjection $\ker v_q \to R^q\pi_*\Omega^1_{\mathcal{X}/S}$, and set $\eta_q := \operatorname{Im} v_{q-1}$. There exists an exact sequence of coherent sheaves on U

$$0 \to \eta_q \xrightarrow{\varphi_q} \xi_q \to (R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor} \to 0$$

such that η_q, ξ_q are locally free with equal rank. Under the canonical isomorphism $\det(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor} \cong \det\xi_q\otimes(\det\eta_q)^{-1}$, the canonical section $\det\varphi_q \in H^0(U,\det\xi_q\otimes(\det\eta_q)^{-1})$ induces the trivialization $\det(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor} \cong \mathcal{O}_U$ on U^o by [50, pp.118, Proof of Lemma 1, First Case]:

$$\det(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor} \ni \det\varphi_q \to 1 \in \mathcal{O}_U. \tag{9.13}$$

Since $\det R^q\pi_*\Omega^1_{\mathcal{X}/S} \cong \det j(R^q\pi_*\Omega^1_{\mathcal{X}/S})\otimes\det(R^q\pi_*\Omega^1_{\mathcal{X}/S})_{\rm tor}$ by (9.10) and (9.11), we deduce from (9.12), (9.13) that the following expression $s_{1,q}$ is a holomorphic section of $\det R^q\pi_*\Omega^1_{\mathcal{X}/S}$:

$$s_{1,q}(t) := (t^{\nu_1}e_1(t)\wedge\cdots\wedge t^{\nu_r}e_r(t))\otimes\det\varphi_q(t).$$

Since $s_{1,q}(t)|_{U^o}$ is identified with the section $t^{\nu_1}e_1(t)\wedge\cdots\wedge t^{\nu_r}e_r(t)|_{U^o}$ under the identification $\det R^q\pi_*\Omega^1_{\mathcal{X}/S}|_{U^o} \cong \det j(R^q\pi_*\Omega^1_{\mathcal{X}/S})|_{U^o}$ induced by (9.13), we deduce from Proposition 9.4 that for $t \in U^o$,

$$\begin{aligned}\log\|s_{1,q}(t)\|^2_{L^2,g_{\mathcal{X}/S}} &= \log\|t^{\nu_1}e_1(t)\wedge\cdots\wedge t^{\nu_r}e_r(t)\|^2_{L^2,g_{\mathcal{X}/S}}\\ &= \dim_{\mathbb{C}}N_q\log|t|^2 + \log\|e_1(t)\wedge\cdots\wedge e_r(t)\|^2_{L^2,g_{\mathcal{X}/S}}\\ &= (\dim_{\mathbb{C}}N_q+\delta_q)\,\log|t|^2 + O(\log(-\log|t|)).\end{aligned} \tag{9.14}$$

Since $\det\varphi_q$ vanishes at $t=0$ with multiplicity $\dim_{\mathbb{C}}M_q$, $\sigma_{1,q}(t) := t^{-\dim_{\mathbb{C}}M_q}\,s_{1,q}(t)$ is a nowhere vanishing holomorphic section of $\det R^q\pi_*\Omega^1_{\mathcal{X}/S}$. By (9.14), we get

$$\log\|\sigma_{1,q}(t)\|^2_{L^2,g_{\mathcal{X}/S}} = (\dim_{\mathbb{C}}N_q+\delta_q-\dim_{\mathbb{C}}M_q)\,\log|t|^2 + O(\log(-\log|t|)). \tag{9.15}$$

The result follows from (9.15). This completes the proof of Proposition 9.5. □

Proposition 9.6. *Let $p=2,3$. There exists $\beta_p\in\mathbb{R}$ such that as $t\to 0$,*

$$\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}=\beta_p\log|t|^2+O(\log(-\log|t|)).$$

Proof. We keep the notation in Section 8, Proof of Theorem 8.1. By Theorem 5.4, there exists $a_p\in\mathbb{Q}$ such that

$$\log\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,Q,g_{\mathcal{X}/S}}=a_p\log|t|^2+O(1).\tag{9.16}$$

By the same argument as in the proof of Theorem 8.1 (8.4) using (9.16) in stead of (8.3), we get

$$\log\|\sigma_p(t)\otimes\sigma_{3-p}(t)^{-1}\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S})\otimes\lambda(\mathcal{E}^{3-p}_{\mathcal{X}/S})^\vee,L^2,g_{\mathcal{X}/S}}=a_p\,\log|t|^2+O(1),$$

which, together with Propositions 9.3 and 9.5, yields the existence of $\beta_p\in\mathbb{R}$ such that

$$\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}=\beta_p\log|t|^2+O(\log(-\log|t|)).$$

This proves the proposition. □

9.2. Proof of Theorem 9.1: the case of semi-stable degenerations

Let γ_t be the Kähler form of $g_{\mathcal{X}/S}|_{X_t}$. By the definition of the BCOV torsion of (X_t,γ_t), we have

$$\log\mathcal{T}_{\mathrm{BCOV}}(X_t,\gamma_t)=\sum_p(-1)^pp\,\{\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),Q,g_{\mathcal{X}/S}}-\log\|\sigma_p(t)\|^2_{\lambda(\mathcal{E}^p_{\mathcal{X}/S}),L^2,g_{\mathcal{X}/S}}\}.$$

By Theorem 5.4 and Propositions 9.3, 9.5, 9.6, there exists $a\in\mathbb{R}$ such that

$$\log\mathcal{T}_{\mathrm{BCOV}}(X_t,\gamma_t)=a\,\log|t|^2+O(\log(-\log|t|^2)).\tag{9.17}$$

Since the Kähler class of $g_{\mathcal{X}}$ is integral, there exist positive constants $A,B\in\mathbb{Q}$ by Lemma 4.12 such that for all $t\in U^o$,

$$\log\mathrm{Vol}(X_t,\gamma_t)=A,\qquad\log\mathrm{Vol}_{L^2}(H^2(X_t,\mathbb{Z}),[\gamma_t])=B.\tag{9.18}$$

By Proposition 7.9 (1), there exists $\epsilon\in\mathbb{R}$ such that

$$\log\mathcal{A}(X_t,\gamma_t)=\epsilon\,\log|t|^2+O(\log(-\log|t|^2)).\tag{9.19}$$

By (9.17), (9.18), (9.19), we get

$$\begin{aligned}\log\tau_{\mathrm{BCOV}}(X_t)&=\log\mathcal{A}(X_t,\gamma_t)+\log\mathcal{T}_{\mathrm{BCOV}}(X_t,\gamma_t)+O(1)\\&=(a+\epsilon)\,\log|t|^2+O(\log(-\log|t|^2)).\end{aligned}$$

This proves the theorem. □

9.3. Proof of Theorem 9.1: general cases

In Subsection 9.3, we only assume that $\pi^o\colon\mathcal{X}^o\to S^o$ is a smooth morphism whose fibers are Calabi-Yau threefolds.

By the semi-stable reduction theorem [27, Chap. II], there exist a pointed projective curve (B,o), a finite surjective holomorphic map $f\colon(B,o)\to(S,0)$, and a holomorphic surjection $p\colon\mathcal{Y}\to B$ from a projective fourfold $\mathcal{Y}$ to B satisfying the following conditions:

(i) Let V be the component of $f^{-1}(U)$ containing o. Then $f\colon V\setminus\{o\}\to U\setminus\{0\}$ is an isomorphism;

(ii) Set $U^*=U\setminus\{0\}$ and $V^*=V\setminus\{o\}$. Then $p|_{V^*}\colon\mathcal{Y}|_{V^*}\to V^*$ is induced from $\pi|_{U^*}\colon\mathcal{X}|_{U^*}\to U^*$ by $f|_{V^*}$;

(iii) $\mathcal{Y}$ is smooth, and Y_o is a reduced divisor of normal crossing.

Let b be the coordinate on V centered at o. By condition (i), we may assume that there exists $\nu \in \mathbb{N}$ such that $f^*t = b^\nu$. Let τ_{U^*} and τ_{V^*} be the functions on U^* and V^* defined by

$$\tau_{U^*}(t) := \tau_{\mathrm{BCOV}}(X_t), \qquad \tau_{V^*}(b) := \tau_{\mathrm{BCOV}}(Y_b)$$

for $t \in U^*$ and $b \in V^*$, respectively. By condition (ii) and Theorem 4.16, we get

$$\tau_{V^*} = f^*\tau_{U^*} \tag{9.20}$$

We can apply Theorem 9.1 to the family $p|_V : \mathcal{Y}|_V \to V$ by condition (iii), so that there exists $\alpha \in \mathbb{R}$ such that as $b \to 0$,

$$\log \tau_{V^*}(b) = \alpha \log |b|^2 + O(\log(-\log |b|)). \tag{9.21}$$

Since $b = t^\nu$, the desired formula follows from (9.20) and (9.21). This completes the proof of Theorem 9.1. □

10. The curvature current of the BCOV invariant

Following [60, Sect. 7], we extend Theorem 4.14 to the Kuranishi space of Calabi-Yau threefold with a unique ODP as its singular set.

10.1. The curvature current of τ_{BCOV}: general cases

In Subsection 10.1, we fix the following notation. Let $\mathcal{X}$ be an irreducible projective algebraic fourfold and let S be a compact Riemann surface. Let $\pi : \mathcal{X} \to S$ be a surjective, flat holomorphic map. Let $\mathcal{D} \subset S$ be a reduced divisor and set $\mathcal{X}^o := \mathcal{X} \setminus \pi^{-1}(\mathcal{D})$, $S^o := S \setminus \mathcal{D}$, $\pi^o := \pi|_{\mathcal{X}^o}$. We assume that the fibers of $\pi^o : \mathcal{X}^o \to S^o$ are Calabi-Yau threefolds with $h^2(\Omega^1_{X_s}) = 1$ for $s \in S^o$. Let $\chi(X)$ denote the topological Euler number of X_s, $s \in S^o$.

Let $\Omega_{\mathrm{WP},\mathcal{X}/S}$ and $\Omega_{\mathrm{H},\mathcal{X}/S}$ be the trivial extensions of the Weil-Petersson form and the Hodge form from S^o to S (cf. Proposition 7.3 and Definition 7.4). Then the $(1,1)$-currents $\Omega_{\mathrm{WP},\mathcal{X}/S}$ and $\Omega_{\mathrm{H},\mathcal{X}/S}$ are positive.

Let $0 \in \mathcal{D}$ and let (U, t) be a coordinate neighborhood of S centered at 0. By Eq. (7.7), there exist subharmonic functions φ and θ on U satisfying the following equations of currents on U:

$$dd^c\varphi = \Omega_{\mathrm{WP},\mathcal{X}/S}|_U, \qquad dd^c\theta = \Omega_{\mathrm{H},\mathcal{X}/S}|_U. \tag{10.1}$$

As in Subsection 4.4.2, we define a function on S by

$$\tau_{\mathrm{BCOV}}(\mathcal{X}/S)(t) := \tau_{\mathrm{BCOV}}(X_t), \qquad t \in S.$$

By Theorems 4.14 and 9.1, $\log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) \in C^\infty(S^o) \cap L^1(S)$.

Theorem 10.1. *Set*

$$a := \lim_{t \to 0} \frac{\log \tau_{\mathrm{BCOV}}(\mathcal{X}/S)|_U(t)}{\log |t|^2} \in \mathbb{R}.$$

Then the following equation of currents on U holds:

$$dd^c \log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) = -\frac{\chi(X)}{12}\,\Omega_{\mathrm{WP},\mathcal{X}/S} - \Omega_{\mathrm{H},\mathcal{X}/S} + a\,\delta_0.$$

Proof. Identify U with Δ in what follows. By Theorem 9.1, there exists a positive constant K such that

$$\left|\log \tau_{\mathrm{BCOV}}(\mathcal{X}/S)(t) - a \log |t|^2\right| \le K \log(-\log |t|), \qquad t \in \Delta(1/2)^*. \tag{10.2}$$

For $t \in \varDelta(1/2)^*$, set

$$P(t) := \left(\log \tau_{\mathrm{BCOV}}(\mathcal{X}/S)(t) - a\,\log|t|^2\right) + \frac{\chi(X)}{12}\,\varphi(t) + \theta(t).$$

Then $P(t) \in C^\infty(\varDelta(1/2)^*)$. By (7.10) and (10.2), there exists a positive constant L such that

$$|P(t)| \le L\,\log(-\log|t|^2), \qquad t \in \varDelta(1/2)^*. \tag{10.3}$$

Since P is harmonic on $\varDelta(1/2)^*$ by Theorem 4.14 and (10.1), we deduce from Lemma 7.1 (3) that P extends to a harmonic function on $\varDelta(1/2)$. Since P is harmonic on $\varDelta(1/2)$, it follows from (7.10) that

$$\log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) = a\,\log|t|^2 - \frac{\chi(X)}{12}\,\varphi - \theta + P \in L^1_{\mathrm{loc}}(\varDelta(1/2)). \tag{10.4}$$

Since $dd^cP = 0$ on $\varDelta$, Eq. (10.4), together with (10.1), yields the assertion. $\square$

10.2. The curvature current of τ_{BCOV}: the case of Kuranishi families

In Subsection 10.2, we fix the following notation: Let X be a smoothable Calabi-Yau threefold with only one ODP as its singular set. Let $\mathrm{Def}(X)$ be the Kuranishi space of X with discriminant locus $\mathfrak{D}$, and let $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X. Assume that $\dim \mathrm{Def}(X) = h^2(\Omega^1_X) = 1$. Let s be a coordinate on $\mathrm{Def}(X)$ such that $\mathfrak{D} = \mathrm{div}(s)$. We identify $\mathrm{Def}(X)$ with the disc $\varDelta$ equipped with the coordinate s. Then $\mathrm{Def}(X) \setminus \mathfrak{D} \cong \varDelta^*$.

Let Ω_{WP} and Ω_{H} be the trivial extensions of the Weil-Petersson form and the Hodge form from $\mathrm{Def}(X) \setminus \mathfrak{D}$ to $\mathrm{Def}(X)$. Let $\chi(X_{\mathrm{gen}})$ denote the topological Euler number of a general fiber of the Kuranishi family.

Theorem 10.2. *The function* $\log \tau_{\mathrm{BCOV}}$ *is locally integrable on* $\mathrm{Def}(X)$*, and the following equation of currents on* $\mathrm{Def}(X)$ *holds:*

$$dd^c \log \tau_{\mathrm{BCOV}} = -\frac{\chi(X_{\mathrm{gen}})}{12}\,\Omega_{\mathrm{WP}} - \Omega_{\mathrm{H}} + \frac{1}{6}\,\delta_{\mathfrak{D}}.$$

Proof. By Proposition 2.8, there exist a pointed projective curve $(B, 0)$, a projective fourfold $\mathfrak{Z}$, and a surjective, proper, flat holomorphic map $f\colon \mathfrak{Z} \to B$ such that the deformation germ $f\colon (\mathfrak{Z}, f^{-1}(0)) \to (B, 0)$ is isomorphic to the Kuranishi family $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$. Since $\mathrm{Def}(X)$ is smooth at $[X]$, so is B at 0. By Theorem 9.1, we get $\log \tau_{\mathrm{BCOV}} \in L^1_{\mathrm{loc}}(\mathrm{Def}(X))$. Let $\gamma := \lim_{t\to 0} \log \tau_{\mathrm{BCOV}}(X_t)/\log|t|^2$. Since $\gamma = \frac{1}{6}$ by Theorem 8.2, the result follows from Theorem 10.1. $\square$

10.3. The curvature current of τ_{BCOV}: the case of induced families

We keep the notation in Subsection 10.2. Let $\mu\colon (\varDelta, 0) \to (\mathrm{Def}(X), [X])$ be a holomorphic map and let $\pi\colon \mathcal{X} \to \varDelta$ be the family of Calabi-Yau threefolds induced from the Kuranishi family $\mathfrak{p}\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ by μ. Notice that $\mathcal{X}$ is singular if 0 is a critical point of μ.

Theorem 10.3. *The function* $\log \tau_{\mathrm{BCOV}}(\mathcal{X}/\varDelta)$ *lies in* $L^1_{\mathrm{loc}}(\varDelta)$*, and the following equation of currents on* $\varDelta$ *holds:*

$$dd^c \log \tau_{\mathrm{BCOV}}(\mathcal{X}/\varDelta) = -\frac{\chi(X_{\mathrm{gen}})}{12}\,\Omega_{\mathrm{WP},\mathcal{X}/\varDelta} - \Omega_{\mathrm{H},\mathcal{X}/\varDelta} + \frac{1}{6}\,\delta_{\mu^*\mathfrak{D}}.$$

Proof. Let $f \in \mathcal{O}_{\mathrm{Def}(X),[X]}$ be such that $\mathfrak{D} = \mathrm{div}(f)$. Let Ω_{WP} and Ω_{H} be the trivial extensions of the Weil-Petersson and the Hodge forms on $\mathrm{Def}(X)$, respectively. As in Eq. (7.7), let φ and θ be the subharmonic functions on $\mathrm{Def}(X)$ with $\Omega_{\mathrm{WP}} = dd^c\varphi$ and $\Omega_{\mathrm{H}} = dd^c\theta$. Then $\mu^*\varphi$ and $\mu^*\theta$ are subharmonic functions on $\varDelta$ with

$$dd^c(\mu^*\varphi)|_{\varDelta^*} = \omega_{\mathrm{WP},\mathcal{X}/\varDelta}, \qquad dd^c(\mu^*\theta)|_{\varDelta^*} = \omega_{\mathrm{H},\mathcal{X}/\varDelta}. \tag{10.5}$$

After shrinking $\mathrm{Def}(X)$ if necessary, we may assume by (7.10) the existence of constants $C_0, C_1 > 0$ with

$$-C_0\,\log(-\log|f|^2) \leq \varphi|_{\mathrm{Def}(X)\setminus\mathfrak{D}} \leq C_1, \qquad -C_0\,\log(-\log|f|^2) \leq \theta|_{\mathrm{Def}(X)\setminus\mathfrak{D}} \leq C_1. \tag{10.6}$$

Since $\mu^{-1}(\mathfrak{D}) \cap \varDelta = \{0\}$, there exist a positive integer k and a nowhere vanishing holomorphic function $\varepsilon(s) \in \mathcal{O}(\varDelta)$ with

$$\mu^* f(s) = s^k\,\varepsilon(s). \tag{10.7}$$

After shrinking $\varDelta$ if necessary, the following inequality holds by (10.6)

$$-C_2\,\log(-\log|s|^2) \leq \mu^*\varphi|_{\varDelta^*} \leq C_1, \qquad -C_2\,\log(-\log|s|^2) \leq \mu^*\theta|_{\varDelta^*} \leq C_1, \tag{10.8}$$

where $C_2 > 0$ is a constant. By (10.5), (10.8) and Lemma 7.5 (2), we get the following equations of currents on $\varDelta$:

$$\Omega_{\mathrm{WP},\mathcal{X}/\varDelta} = dd^c(\mu^*\varphi), \qquad \Omega_{\mathrm{H},\mathcal{X}/\varDelta} = dd^c(\mu^*\theta). \tag{10.9}$$

By (10.4) and Theorem 10.2, there exists a harmonic function P on $\mathrm{Def}(X)$ such that

$$\log\tau_{\mathrm{BCOV}} = \frac{1}{6}\,\log|f|^2 - \frac{\chi(X)}{12}\,\varphi - \theta + P.$$

Since $\tau_{\mathrm{BCOV}}(\mathcal{X}/\varDelta) = \mu^*\tau_{\mathrm{BCOV}}$, we get

$$\log\tau_{\mathrm{BCOV}}(\mathcal{X}/\varDelta) = \frac{1}{6}\,\mu^*\log|f|^2 - \frac{\chi(X)}{12}\,\mu^*\varphi - \mu^*\theta + \mu^*P. \tag{10.10}$$

By (10.8), (10.10), we get $\log\tau_{\mathrm{BCOV}}(\mathcal{X}/\varDelta) \in L^1_{\mathrm{loc}}(\varDelta)$. By (10.9), (10.10), we get the desired equation of currents. This complete the proof. □

11. The BCOV invariant of Calabi-Yau threefolds with $h^{1,2} = 1$

In Section 11, we fix the following notation. Let $\mathcal{X}$ be a possibly singular irreducible projective fourfold and let S be a compact Riemann surface. Let $\pi\colon \mathcal{X} \to S$ be a proper, surjective, flat morphism with discriminant locus $\mathcal{D} := \{s \in S;\ \mathrm{Sing}\,X_s \neq \emptyset\}$. We set

$$S^o := S \setminus \mathcal{D}, \qquad \mathcal{X}^o := \pi^{-1}(S^o),$$

$$\mathcal{D}^* := \{s \in \mathcal{D};\ \mathrm{Sing}\,X_s \text{ consists of a unique ODP}\},$$

and

$$S^* := S^o \cup \mathcal{D}^*, \qquad \mathcal{X}^* := \pi^{-1}(S^*).$$

In Section 11, we make the following:

Assumption (i) X_s is a Calabi-Yau threefold with $h^2(\Omega^1_{X_s}) = 1$ for all $s \in S^*$;
(ii) $\mathcal{D}^*$ is a non-empty finite set, and $\mathcal{D} \setminus \mathcal{D}^*$ consists of a unique point $\infty \in S$;
(iii) $\mathrm{Sing}(\mathcal{X}) \cap X_\infty = \emptyset$ and X_∞ is a divisor of normal crossing.

Lemma 11.1. *Let $p \in \mathcal{D}^*$. Then X_p is smoothable in the sense of Definition* 2.2.

Proof. To see this, let $o = \mathrm{Sing}\, X_p$, and let $f\colon \widetilde{X}_p \to X_p$ be a small resolution such that $C := f^{-1}(o) \cong \mathbb{P}^1$ and $\widetilde{X}_p \setminus C \cong X_p \setminus \{o\}$. Let $[C] \in H_2(\widetilde{X}_p, \mathbb{Z})$ be the homology class of C. Since X_p is smoothable by a flat deformation by Assumption (ii), we get $[C] = 0$ by [42, Th. 2.5 (2)$\Rightarrow$(3)]. Hence the map γ' in [41, p.16, l.28] is zero. By the commutative diagram [41, p.16 (14)], the natural map $\mathrm{Ext}^1(\Omega^1_{X_p}, \mathcal{O}_{X_p}) \to H^0(X, \mathcal{E}xt(\Omega^1_{X_p}, \mathcal{O}_{X_p}))$ is not zero. Let $\mathrm{Def}(X_p, o)$ be the Kuranishi space of the ODP (X_p, o) and let $\phi\colon (\mathrm{Def}_{[X_p]}(X_p), [X_p]) \to (\mathrm{Def}(X_p, o), o)$ be the map of germs induced from the Kuranishi family of X_p. Since $\mathrm{Ext}^1(\Omega^1_{X_p}, \mathcal{O}_{X_p}) = T_{[X_p]}\mathrm{Def}(X_p)$ and $H^0(X, \mathcal{E}xt(\Omega^1_{X_p}, \mathcal{O}_{X_p})) = T_p\mathrm{Def}(X_p, o)$ via the Kodaira-Spencer map and since the natural map $\mathrm{Ext}^1(\Omega^1_{X_p}, \mathcal{O}_{X_p}) \to H^0(X, \mathcal{E}xt(\Omega^1_{X_p}, \mathcal{O}_{X_p}))$ is identified with the differential of ϕ at $[X_p]$, we get $(d\phi)_{[X_p]} \neq 0$. Since $\dim T_{[X_p]}\mathrm{Def}(X_p) = \dim T_o\mathrm{Def}(X_p, o) = 1$ by Assumption (i), $(d\phi)_{[X_p]}$ is an isomorphism. By [41, Prop. 5.3] and the smoothness of $\mathrm{Def}(X_p, o)$, ϕ is an isomorphism of germs. This implies the smoothness of the total space of the Kuranishi family of X_p. $\square$

The *ramification divisor* of the family $\pi\colon \mathcal{X} \to S$ is defined as follows. For $s \in S^*$, let $\mu_s\colon (S, s) \to (\mathrm{Def}(X_s), [X_s])$ be the map of germs of analytic sets defined by

$$\mu_s(t) := [X_t] \in \mathrm{Def}(X_s).$$

By Lemmas 2.7 and 11.1, μ_p is not a constant map for $p \in \mathcal{D}^*$. Since $\mathcal{D}^* \neq \emptyset$ by Assumption (ii), μ_s is not constant for all $s \in S^*$. Since $\dim \mathrm{Def}(X_s) = 1$, we may identify $(\mathrm{Def}(X_s), [X_s])$ with $(\mathbb{C}, 0)$. Let z be the coordinate of $\mathbb{C}$, so that $z \circ \mu_s(t) \in \mathcal{O}_{S,s}$. We define the ramification index of $\pi\colon \mathcal{X} \to S$ at $s \in S$ by

$$r_{\mathcal{X}/S}(s) := \mathrm{ord}_{t=s} z \circ \mu_s(t) \in \mathbb{N}.$$

Let $\{R_j\}_{j \in J}$ be the set of points of S whose ramification index is > 1. The ramification divisor is then defined as

$$\mathcal{R} := \sum_{j \in J} (r_j - 1)\, R_j, \qquad r_j := r_{\mathcal{X}/S}(R_j).$$

Let $p \in \mathcal{D}^*$ and $\mathrm{Sing}(X_p) = \{o\}$. By the local description (2.2), we have an isomorphism of local rings

$$\mathcal{O}_{\mathcal{X},o} \cong \mathbb{C}\{x, y, z, w, t\}/(x^2 + y^2 + z^2 + w^2 + t^{r_{\mathcal{X}/S}(p)}). \tag{11.1}$$

Write $\mathcal{D}^* = \{D_k\}_{k \in K}$. As a divisor of S, we define

$$\mathcal{D}^* := \sum_{k \in K} r_k\, D_k, \qquad r_k := r_{\mathcal{X}/S}(D_k).$$

Since $\mathrm{Sing}\, \mathcal{X} \subset \cup_{s \in \mathcal{D}^*} \mathrm{Sing}\, X_s$, $\mathcal{X}$ has at most isolated hypersurface singularities as its singular points by (11.1). Hence $K_{\mathcal{X}}$ and $K_{\mathcal{X}/S} := K_{\mathcal{X}} \otimes \pi^* K_S^{-1}$ are invertible sheaves on $\mathcal{X}$.

Lemma 11.2. *The sheaf $\pi_* K_{\mathcal{X}/S}$ is an invertible sheaf on S.*

Proof. Since $\pi^{-1}(S \setminus \mathcal{D}^*)$ is smooth, $\pi_* K_{\mathcal{X}/S}$ is an invertible sheaf on $S \setminus \mathcal{D}^*$ by Assumption (i) and [52, p.391, Th. V]. Let $s \in \mathcal{D}^*$. Since the conormal bundle of $(X_s)_{\mathrm{reg}}$ in $\mathcal{X}_{\mathrm{reg}}$ is trivial, we have $K_{\mathcal{X}/S}|_{(X_s)_{\mathrm{reg}}} \cong K_{(X_s)_{\mathrm{reg}}}$. Since $K_{\mathcal{X}/S}|_{X_s}$ and K_{X_s} are invertible sheaves on X_s, we get $K_{\mathcal{X}/S}|_{X_s} \cong K_{X_s}$ by the normality of X_s. Since X_s is Calabi-Yau, we have $h^0(K_{\mathcal{X}/S}|_{X_s}) = h^0(K_{X_s}) = 1$. By [1, Th. 4.12 (ii)], $\pi_* K_{\mathcal{X}/S}$ is an invertible sheaf near $s \in \mathcal{D}^*$. This proves the lemma. $\square$

Let χ be the topological Euler number of a general fiber X_s, $s \in S^o$. Let $\|\cdot\|$ be the Hermitian metric on $(\pi_* K_{\mathcal{X}/S})^{\otimes(48+\chi)} \otimes (TS)^{\otimes 12}|_{S^o}$ induced from the L^2-metric on $\pi_* K_{\mathcal{X}/S}$ and from the Weil-Petersson metric $g_{\mathrm{WP},\mathcal{X}/S}$ on S^o. The following is the main result of this paper.

Main Theorem 11.3. *Let Ξ be a meromorphic section of $\pi_* K_{\mathcal{X}/S}$ on S with*

$$\mathrm{div}(\Xi) = \sum_{i \in I} m_i\, P_i + m_\infty P_\infty, \qquad P_i \neq P_\infty\, (i \in I),$$

and let V be a meromorphic vector field on S. Then the following hold:
(1) *There exists a locally integrable function $F_{\Xi,V}$ on S with*

$$dd^c F_{\Xi,V} = \left\{ (24 + \frac{\chi}{2}) \deg \pi_* K_{\mathcal{X}/S} + 6\, \chi(S) + 6 \deg \mathcal{R} - \deg \mathcal{D}^* \right\} \delta_\infty + \delta_{\mathcal{D}^*} - (24 + \frac{\chi}{2})\, \delta_{\mathrm{div}(\Xi)} - 6\, \delta_{\mathrm{div}(V)} - 6\, \delta_{\mathcal{R}}$$

such that

$$\tau_{\mathrm{BCOV}}(\mathcal{X}/S) = \left\| e^{F_{\Xi,V}}\, \Xi^{48+\chi} \otimes V^{12} \right\|^{\frac{1}{6}}.$$

(2) *When $S = \mathbb{P}^1$, let ψ be the inhomogeneous coordinate of $\mathbb{P}^1$ with $\psi(\infty) = \infty$. Identify the points P_i, R_j, D_k with their coordinates $\psi(P_i), \psi(R_j), \psi(D_k)$, respectively. Then there exists a constant $C \neq 0$ such that*

$$\tau_{\mathrm{BCOV}}(X_\psi) = C \left\| \prod_{i \in I, j \in J, k \in K} \frac{(\psi - D_k)^{2 r_k}}{(\psi - P_i)^{(48+\chi) m_i} (\psi - R_j)^{12(r_j - 1)}}\, \Xi_\psi^{48+\chi} \otimes \left(\frac{\partial}{\partial \psi} \right)^{12} \right\|^{\frac{1}{6}}.$$

In the rest of this section, we shall prove Theorem 11.3. For $p \in \mathcal{D}$, let (U_p, t) be a coordinate neighborhood of S centered at p with $U_p \cap \mathcal{D} = \{p\}$ and $U_p \setminus \{p\} \cong \Delta^*$.

By Proposition 7.3, the positive $(1,1)$-forms $\omega_{\mathrm{WP},\mathcal{X}/S}$ and $\omega_{\mathrm{H},\mathcal{X}/S}$ on S^o extend trivially to closed positive $(1,1)$-currents on S.

Definition 11.4. Let $\Omega_{\mathrm{WP},\mathcal{X}/S}$ and $\Omega_{\mathrm{H},\mathcal{X}/S}$ be the trivial extensions of $\omega_{\mathrm{WP},\mathcal{X}/S}$ and $\omega_{\mathrm{H},\mathcal{X}/S}$ from S^o to S, respectively.

Proposition 11.5. (1) *There exists $a(p) \in \mathbb{R}$ such that the following equation of currents on U_p holds:*

$$dd^c \log \Omega_{\mathrm{WP},\mathcal{X}/S}|_{U_p} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}} \right) = a(p)\, \delta_p - \Omega_{\mathrm{H},\mathcal{X}/S} + 4\, \Omega_{\mathrm{WP},\mathcal{X}/S}.$$

(2) *For $D_k \in \mathcal{D}^*$, one has $a(D_k) = r_k - 1$.*

Proof. We get (1) by Proposition 7.6 (2). Let $p = D_k$. Under the identification of the Kuranishi space $(\mathrm{Def}(X_p), [X_p])$ with $(\mathbb{C}, 0)$, we may assume by the definition of the ramification index $r_{\mathcal{X}/S}$ that $\pi|_{U_p} \colon \mathcal{X}|_{U_p} \to U_p$ is induced from the Kuranishi family of X_p by the map $\mu(t) = t^{r_k}$. Let ω_{WP} be the Weil-Petersson form on $\mathrm{Def}(X_p)$. Since $\Omega_{\mathrm{WP},\mathcal{X}/S}|_{U_p \setminus \{p\}} = \mu^* \omega_{\mathrm{WP}}$, we deduce from Proposition 7.6 (1), (3) that as $t \to 0$,

$$\begin{aligned} \log \Omega_{\mathrm{WP},\mathcal{X}/S}|_{U_p} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}} \right) &= \log \omega_{\mathrm{WP}} \left(\mu_* \frac{\partial}{\partial t}, \mu_* \frac{\partial}{\partial \bar{t}} \right) \\ &= (r_k - 1) \log |t|^2 + O(\log(-\log |t|)). \end{aligned} \tag{11.2}$$

By (11.2), we get $a(p) = r_k - 1$. This completes the proof. ∎

Proposition 11.6. *There exists $b(\infty) \in \mathbb{R}$ such that the following equation of currents on S holds:*

$$dd^c \log \|\Xi\|_{L^2}^2 = b(\infty)\,\delta_\infty + \delta_{\mathrm{div}(\Xi)} - \Omega_{\mathrm{WP},\mathcal{X}/S}. \tag{11.3}$$

Proof. Let $s \in S$ be an arbitrary point. It suffices to prove Eq. (11.3) on a neighborhood of s. By Proposition 7.8 (2), Eq. (11.3) holds on a neighborhood of ∞.

Assume that $s \in S^*$. Let $\mathfrak{p}\colon (\mathfrak{X}, X_s) \to (\mathrm{Def}(X_s), [X_s])$ be the Kuranishi family of X_s. Since $\pi\colon (\mathcal{X}, X_s) \to (S, s)$ is induced from the Kuranishi family by the map $\mu_s\colon (S, s) \to (\mathrm{Def}(X_s), [X_s])$, there exists a morphism of deformation germs $f_{\mu_s}\colon (\mathcal{X}, X_s) \to (\mathfrak{X}, X_s)$ satisfying the commutative diagram:

$$\begin{array}{ccc} (\mathcal{X}, X_s) & \xrightarrow{f_{\mu_s}} & (\mathfrak{X}, X_s) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\mathfrak{p}}\big\downarrow \\ (S, s) & \xrightarrow{\mu_s} & (\mathrm{Def}(X_s), [X_s]). \end{array}$$

Let $U_s \cong \Delta$ be a neighborhood of s in S such that μ_s (resp. f_{μ_s}) is defined on U_s (resp. $\pi^{-1}(U_s)$) and such that μ_s has no critical points on $U_s^o := U_s \setminus \{s\}$. Since

$$f_{\mu_s}^* K_{\mathfrak{X}/\mathrm{Def}(X_s)} = K_{\mathcal{X}/S} \tag{11.4}$$

on $\pi^{-1}(U_s) \setminus \mathrm{Sing}\, X_s$, the normality of $\mathcal{X}$ implies that (11.4) holds on $\pi^{-1}(U_s)$.

By Lemma 6.1, $K_{\mathfrak{X}/\mathrm{Def}(X_s)}$ is trivial. Let $\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}$ be a nowhere vanishing holomorphic section of $K_{\mathfrak{X}/\mathrm{Def}(X_s)}$ defined on $\mathrm{Def}(X_s)$. We regard $\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}$ as a family of holomorphic 3-forms $\{\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}|_{\mathfrak{X}_b}\}_{b \in \mathrm{Def}(X_s)}$. Since X_s has at most one ODP as its singular set, $\log \|\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2} \in C^0(\mathrm{Def}(X_s))$ by Proposition 7.8 (3).

Since $f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)} \in H^0(\pi^{-1}(U_s), K_{\mathcal{X}/S}) = H^0(U_s, \pi_* K_{\mathcal{X}/S})$ is nowhere vanishing, $f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}$ generates $\pi_* K_{\mathcal{X}/S}$ on U_s as an $\mathcal{O}_{U_s}$-module. Since

$$\|f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2}(t) = \|\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2}(\mu_s(t)), \qquad t \in U_s^o$$

by (11.4) and since $\log \|\eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2} \in C^0(\mathrm{Def}(X_s))$, $\log \|f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2}$ is a continuous function on U_s. Since $-dd^c \log \|f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2} = \Omega_{\mathrm{WP},\mathcal{X}/S}$ on U_s^o, we get the following equation of currents on U_s by Lemma 7.5 (1), (2):

$$-dd^c \log \|f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}\|_{L^2} = \Omega_{\mathrm{WP},\mathcal{X}/S}. \tag{11.5}$$

Since $f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)} \in H^0(U_s, \pi_* K_{\mathcal{X}/S})$ is nowhere vanishing, there exists $h(t) \in \mathcal{O}(U_s)$ such that $\Xi = h \cdot f_{\mu_s}^* \eta_{\mathfrak{X}/\mathrm{Def}(X_s)}$ on U_s. By (11.5), we get

$$-dd^c \log \|\Xi\|_{L^2}^2 = \Omega_{\mathrm{WP},\mathcal{X}/S} - \delta_{\mathrm{div}(h)} \tag{11.6}$$

as currents on U_s. Eq. (11.3) on U_s follows from (11.6). □

Theorem 11.7. *There exists $c(\infty) \in \mathbb{Q}$ such that the following equation of currents on S holds:*

$$dd^c \log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) = -\frac{\chi}{12}\,\Omega_{\mathrm{WP},\mathcal{X}/S} - \Omega_{\mathrm{H},\mathcal{X}/S} + \frac{1}{6}\,\delta_{\mathcal{D}^*} + c(\infty)\,\delta_\infty. \tag{11.7}$$

Proof. The result follows from Theorems 10.1 and 10.3. □

Proof of Theorem 11.3 (1) By Proposition 11.5 and (4.1), we get the following equation of currents on S:

$$dd^c \log \|V\|^2 = a(\infty)\,\delta_\infty + \delta_{\mathcal{R}} + \delta_{\mathrm{div}(V)} - \Omega_{\mathrm{H},\mathcal{X}/S} + 4\,\Omega_{\mathrm{WP},\mathcal{X}/S}. \tag{11.8}$$

By (11.3), (11.7), (11.8), we get

(11.9)
$$
\begin{aligned}
dd^c \log \|V^{12} \otimes \Xi^{48+\chi}\|^2 &= 12(a(\infty)\,\delta_\infty + \delta_{\mathcal{R}} + \delta_{\mathrm{div}(V)}) - 12\,\Omega_{\mathrm{H},\mathcal{X}/S} + 48\,\Omega_{\mathrm{WP},\mathcal{X}/S} \\
&\quad + (48+\chi)\,(b(\infty)\,\delta_\infty + \delta_{\mathrm{div}(\Xi)}) - (48+\chi)\,\Omega_{\mathrm{WP},\mathcal{X}/S} \\
&= 12\,dd^c \log \tau_{\mathrm{BCOV}}(\mathcal{X}/S) \\
&\quad + \{12\,a(\infty) + (48+\chi)\,b(\infty) - 12\,c(\infty)\}\,\delta_\infty \\
&\quad - 2\,\delta_{\mathcal{D}^*} + 12\,\delta_{\mathcal{R}} + 12\,\delta_{\mathrm{div}(V)} + (48+\chi)\,\delta_{\mathrm{div}(\Xi)}.
\end{aligned}
$$

Integrating the both hand sides of (11.9) over S, we get

(11.10)
$$
\{12\,a(\infty)+(48+\chi)\,b(\infty)-12\,c(\infty)\}-2\,\deg \mathcal{D}^*+12\,\deg \mathcal{R}+12\,\chi(S)+(48+\chi)\,\deg \Xi = 0.
$$

By (11.9) and (11.10),

$$
F_{\Xi,V} := \log \tau_{\mathrm{BCOV}}(\mathcal{X}/S)^6 - \log \|V^{12} \otimes \Xi^{48+\chi}\|
$$

is a harmonic function on $S \setminus (\mathcal{D} \cup \mathcal{R})$ satisfying Theorem 11.3 (1). This proves (1).
(2) We set $V(\psi) := \partial/\partial\psi \in H^0(\mathbb{P}^1, T\mathbb{P}^1)$. Then $\mathrm{div}(V) = 2\,\infty$, so that $F_{\Xi,V}$ satisfies the following equation of currents on $\mathbb{P}^1$ by (11.9), (11.10):

$$
\begin{aligned}
dd^c F_{\Xi,V} &= \left\{(24 + \frac{\chi}{2})\,\deg \pi_* K_{\mathcal{X}/S} + 6\,\deg \mathcal{R} - \deg \mathcal{D}^*\right\}\,\delta_\infty \\
&\quad + \delta_{\mathcal{D}^*} - (24 + \frac{\chi}{2})\,\delta_{\mathrm{div}(\Xi)} - 6\,\delta_{\mathcal{R}}.
\end{aligned}
\tag{11.11}
$$

Up to a constant, the solution of Eq. (11.11) is given by the following formula:

$$
F_{\Xi,V}(\psi) = \log \left| \prod_{i \in I, j \in J, k \in K} \frac{(\psi - D_k)^{2r_k}}{(\psi - P_i)^{(48+\chi)m_i}(\psi - R_j)^{12(r_j-1)}} \right|.
\tag{11.12}
$$

The second assertion of Theorem 11.3 follows from (11.12). This completes the proof of Theorem 11.3. □

12. **The BCOV invariant of quintic mirror threefolds**

12.1. **Quintic mirror threefolds**

Let $p\colon \mathcal{X} \to \mathbb{P}^1$ be the pencil of quintic threefolds in $\mathbb{P}^4$ defined by

$$
\mathcal{X} := \{([z], \psi) \in \mathbb{P}^4 \times \mathbb{P}^1;\ F_\psi(z) = 0\}, \qquad p = \mathrm{pr}_2,
$$

$$
F_\psi(z) := z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4.
$$

The parameter ψ is regarded as the inhomogeneous coordinate of $\mathbb{P}^1$. Identify $\mathbb{Z}_5$ with the set of fifth roots of unity: $\mathbb{Z}_5 = \{\zeta \in \mathbb{C};\ \zeta^5 = 1\}$. We define

$$
G := \frac{\{(a_0, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_5)^5;\ a_0 a_1 a_2 a_3 a_4 = 1\}}{\mathbb{Z}_5(1,1,1,1,1)} \cong \mathbb{Z}_5^3.
$$

The group $G \times \mathbb{Z}_5$ acts on $\mathcal{X}$ and $\mathbb{P}^1$ by

$$
(a,b) \cdot ([z], \psi) := ([b^{-1} a_0 z_0 : a_1 z_1 : a_2 z_2 : a_3 z_3 : a_4 z_4], b\psi), \qquad (a,b) \cdot \psi := b\,\psi.
$$

Then the projection $p\colon \mathcal{X} \to \mathbb{P}^1$ is $G \times \mathbb{Z}_5$-equivariant. Since G preserves the fibers of p, we have the induced family

$$
p\colon \mathcal{X}/G \to \mathbb{P}^1
$$

equipped with the induced $\mathbb{Z}_5$-action. We set

$$\mathcal{D}^* := \left\{\exp\frac{2\pi\sqrt{-1}\,m}{5} \in \mathbb{P}^1;\ 0 \leq m \leq 4\right\} \subset \mathbb{P}^1, \qquad \mathcal{D} := \mathcal{D}^* \cup \{\infty\} \subset \mathbb{P}^1.$$

Then $\mathcal{D}$ is the discriminant locus of the family $p\colon \mathcal{X} \to \mathbb{P}^1$ by [14, p.27].

Proposition 12.1. *There exists a resolution $f\colon \mathcal{W} \to \mathcal{X}/G$ satisfying the following conditions:*
(1) *Set $f_\psi := f|_{W_\psi}$. Then $f_\psi\colon W_\psi \to X_\psi/G$ is a crepant resolution for $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$. In particular, W_ψ is a smooth Calabi-Yau threefold for $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$;*
(2) $\operatorname{Sing} W_\psi$ *consists of a unique ODP if $\psi^5 = 1$;*
(3) *W_∞ is a divisor of normal crossing.*

Proof. See [39, Appendix B], [4], [15, Sects. 2.2 and 2.4] for (1) and [14, p.27] for (2). The last assertion follows from Hironaka's theorem. □

Notice that the choice of a resolution $f\colon \mathcal{W} \to \mathcal{X}/G$ as above is *not* unique.

Definition 12.2. Set $\pi := p \circ f$. Any family $\pi\colon \mathcal{W} \to \mathbb{P}^1$ satisfying the conditions (1), (2), (3) as above is called a *family of quintic mirror threefolds*. The induced family $\pi\colon \mathcal{W}/\mathbb{Z}_5 \to \mathbb{P}^1/\mathbb{Z}_5$ is also called a family of quintic mirror threefolds.

Lemma 12.3. *If $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$, then*

$$h^{1,2}(W_\psi) = 1, \qquad h^{1,1}(W_\psi) = 101, \qquad \chi(W_\psi) = 200.$$

Proof. Since $h^{1,1}(X_\psi) = 1$, $h^{1,2}(X_\psi) = 101$, and $\chi(X_\psi) = -200$, the result follows from [4], [15, Th. 4.1.5], [56, Th. 4.30]. □

We refer to [14], [15], [39], [56] for more details about quintic mirror threefolds.

12.2. The mirror map

Definition 12.4. The *mirror map* is the holomorphic map from a neighborhood of $\infty \in \mathbb{P}^1$ to a neighborhood of $0 \in \Delta$ defined by the following formula:

$$q := (5\psi)^{-5}\exp\left(\frac{5}{y_0(\psi)}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left\{\sum_{j=n+1}^{5n}\frac{1}{j}\right\}\frac{1}{(5\psi)^{5n}}\right), \qquad |\psi| \gg 1,$$

where

$$y_0(\psi) := \sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5(5\psi)^{5n}}, \qquad |\psi| > 1.$$

The inverse of the mirror map is denoted by $\psi(q)$.

For $\psi \in \mathbb{P}^1 \setminus \mathcal{D}$, we define a holomorphic 3-form on X_ψ by

$$\varOmega_\psi := \left(\frac{2\pi\sqrt{-1}}{5}\right)^{-3} 5\psi\frac{z_4\,dz_0 \wedge dz_1 \wedge dz_2}{\partial F_\psi(z)/\partial z_3}.$$

Since $\varOmega_\psi$ is G-invariant, $\varOmega_\psi$ induces a holomorphic 3-form on X_ψ/G in the sense of orbifolds. We identify $\varOmega_\psi$ with the corresponding holomorphic 3-form on X_ψ/G. Then $f_\psi^*\varOmega_\psi$ is a holomorphic 3-form on W_ψ. Set

$$\varXi_\psi := f_\psi^*\varOmega_\psi \in H^0(W_\psi, K_{W_\psi}).$$

By Lemma 12.3, we know $\operatorname{rk} H_3(W_\psi, \mathbb{Z}) = 4$. There exists a symplectic basis $\{A^1, A^2, B_1, B_2\}$ of $H_3(W_\psi, \mathbb{Q})$, $\psi \notin \mathcal{D}$, such that $A^a \cap B_b = \delta_{an}$, $A^a \cap A^b = B_a \cap B_b = 0$. By [14], [39, p.245 l.13], the mirror map $q(\psi)$ is expressed as follows:

$$q = \exp\left(2\pi\sqrt{-1}\,\frac{\int_{2B_1 - A^1} \Xi_\psi}{\int_{A^2} \Xi_\psi}\right), \qquad y_0(\psi) = \int_{A^2} \Xi_\psi.$$

We refer to [14], [15, Sect. 2.3, Sect. 6.3], [39], [56, Chap. 3] for more details about the mirror map.

12.3. Conjectures of Bershadsky-Cecotti-Ooguri-Vafa

Definition 12.5. Under the identification of the local parameters ψ^5 and q via the mirror map, define a multi-valued analytic function near $\infty \in \mathbb{P}^1$ as

$$F_{1,B}^{\mathrm{top}}(\psi) := \left(\frac{\psi}{y_0(\psi)}\right)^{\frac{62}{3}} (\psi^5 - 1)^{-\frac{1}{6}}\, q\,\frac{d\psi}{dq}$$

and a power series in q as

$$F_{1,A}^{\mathrm{top}}(q) := F_{1,B}^{\mathrm{top}}(\psi(q)).$$

Set

$$\eta(q) := \prod_{n=1}^{\infty} (1 - q^n).$$

In [6, Eq.(16), (23), (24)] and [7, l.34], Bershadsky-Cecotti-Ooguri-Vafa conjectured the following:

Conjecture 12.6. (A) *Let $N_g(d)$ be the genus-g Gromov-Witten invariant of degree d of a general quintic threefold in $\mathbb{P}^4$ (cf. [34]). Then the following identity holds:*

$$q\,\frac{d}{dq}\log F_{1,A}^{\mathrm{top}}(q) = \frac{50}{12} - \sum_{n,d=1}^{\infty} N_1(d)\,\frac{2nd\,q^{nd}}{1 - q^{nd}} - \sum_{d=1}^{\infty} N_0(d)\,\frac{2d\,q^d}{12(1 - q^d)},$$

or equivalently

$$F_{1,A}^{\mathrm{top}}(q) = \mathrm{Const.}\left\{q^{25/12}\prod_{d=1}^{\infty}\eta(q^d)^{N_1(d)}(1 - q^d)^{N_0(d)/12}\right\}^2.$$

(B) *Let $\|\cdot\|$ be the Hermitian metric on the line bundle $(\pi_* K_{\mathcal{W}/\mathbb{P}^1})^{\otimes 62} \otimes (T\mathbb{P}^1)^{\otimes 3}|_{\mathbb{P}^1 \setminus \mathcal{D}}$ induced from the L^2-metric on $\pi_* K_{\mathcal{W}/\mathbb{P}^1}$ and from the Weil-Petersson metric on $T\mathbb{P}^1$. Then the following identity holds:*

$$\begin{aligned}\tau_{\mathrm{BCOV}}(W_\psi) &= \mathrm{Const.}\left\|\psi^{-62}(\psi^5 - 1)^{\frac{1}{2}}\,(\Xi_\psi)^{62} \otimes \left(\frac{d}{d\psi}\right)^3\right\|^{\frac{2}{3}} \\ &= \mathrm{Const.}\left\|\frac{1}{F_{1,B}^{\mathrm{top}}(\psi)^3}\,\left(\frac{\Xi_\psi}{y_0(\psi)}\right)^{62} \otimes \left(q\frac{d}{dq}\right)^3\right\|^{\frac{2}{3}}.\end{aligned}$$

Remark 12.7. Under Conjecture 12.6, the Gromov-Witten invariants $\{N_g(d)\}_{g\leq 1, d\in\mathbb{N}}$ of a general quintic threefold in $\mathbb{P}^4$ and the BCOV invariant of the mirror quintic threefolds satisfy the following relation:

$$\tau_{\rm BCOV}(W_\psi) = \text{Const.} \left\| \left\{ q^{\frac{25}{12}} \prod_{d=1}^{\infty} \eta(q^d)^{N_1(d)}(1-q^d)^{\frac{N_0(d)}{12}} \right\}^6 \left(\frac{\Xi_\psi}{y_0(\psi)}\right)^{62} \otimes \left(q\frac{d}{dq}\right)^3 \right\|^{\frac{2}{3}}.$$

In the rest of this section, we prove Conjecture 12.6 (B) as an application of Theorem 11.3.

12.4. **Proof of Conjecture 12.6 (B)**

Let $\pi\colon \mathcal{W} \to \mathbb{P}^1$ be a family of quintic mirror threefolds. Let $K(\psi)$ be the Kähler potential of the Weil-Petersson form $\Omega_{\rm WP}$ defined as

$$K(\psi) := -\log\left(\sqrt{-1}\int_{W_\psi} \Xi_\psi \wedge \overline{\Xi}_\psi\right).$$

Define a function $G(\psi)$ by $G(\psi) = g_{\rm WP}(\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\psi})$, so that

$$\Omega_{\rm WP}(\psi) = \sqrt{-1}\,G(\psi)\,d\psi\wedge d\bar\psi = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2 K(\psi)}{\partial\psi\partial\bar\psi}\,d\psi\wedge d\bar\psi.$$

Proposition 12.8. *The following estimates hold*

$$K(\psi) = \begin{cases} \log|\psi|^2 + O(1) & (\psi\to 0) \\ O(1) & (\psi^5 \to 1) \\ O(\log\log|\psi|) & (\psi\to\infty), \end{cases} \tag{12.1}$$

$$\log G(\psi) = \begin{cases} O(1) & (\psi\to 0) \\ O(\log(-\log|\psi^5-1|)) & (\psi^5\to 1) \\ -\log|\psi|^2 + O(\log\log|\psi|) & (\psi\to\infty). \end{cases} \tag{12.2}$$

In particular, $\mathcal{R}\cap\mathcal{D}^* = \emptyset$ for any family of quintic mirror threefolds.

Proof. See [14, p.50 Table 2]. □

Proposition 12.9. *The family of quintic mirror threefolds has trivial ramification divisor, i.e., $\mathcal{R} = 0$ for the family $\pi\colon \mathcal{W}\to\mathbb{P}^1$.*

Proof. By (11.2) and Proposition 12.8, if suffices to prove that $G(\psi) > 0$ on $\mathbb{P}^1\setminus\mathcal{D}$. Since

$$K(\psi) = -\log\left(\frac{\sqrt{-1}}{|G|}\int_{X_\psi}\Omega_\psi\wedge\overline{\Omega}_\psi\right),$$

$\Omega_{\rm WP}(\psi)$ coincides with the Weil-Petersson form for X_ψ by (4.1). Thus $G(\psi)>0$ if and only if the Kodaira-Spencer map $\mu_\psi\colon T_\psi\mathbb{P}^1 \to H^1(X_\psi, \Theta_{X_\psi})$ for $p\colon \mathcal{X}\to\mathbb{P}^1$ is non-degenerate at $\psi\in\mathbb{P}^1\setminus\mathcal{D}$. By [56, p.53 l.18-l.27], μ_ψ is non-degenerate for all $\psi\in\mathbb{P}^1\setminus\mathcal{D}$. This proves the proposition. □

Theorem 12.10. *Conjecture 12.6 (B) holds.*

Proof. For a point $z = (1 : z) \in \mathbb{P}^1$, let $[z] = [(1 : z)]$ denote the corresponding divisor. By Proposition 12.1, we get

$$\operatorname{div}(\mathcal{D}^*) = \sum_{\zeta^5=1} [\zeta], \tag{12.3}$$

which is a reduced divisor. By (12.1), we have

$$\operatorname{div}(\Xi) = [0]. \tag{12.4}$$

Substituting (12.3), (12.4) and $\mathcal{R} = 0$ into the formula for τ_{BCOV} in Theorem 11.3 (2) and using $\chi(W_\psi) = 200$, we get

$$\begin{aligned} \tau_{\mathrm{BCOV}}(W_\psi) &= \mathrm{Const.} \left\| \frac{\prod_{\zeta^5=1}(\psi - \zeta)^2}{\psi^{48+\chi}}\, \Xi_\psi^{48+\chi} \otimes \left(\frac{\partial}{\partial\psi}\right)^{12} \right\|^{1/6} \\ &= \mathrm{Const.} \left\| \frac{(\psi^5 - 1)^2}{\psi^{248}}\, \Xi_\psi^{248} \otimes \left(\frac{\partial}{\partial\psi}\right)^{12} \right\|^{1/6} \\ &= \mathrm{Const.} \left\| \psi^{-62}(\psi^5 - 1)^{1/2}\, \Xi_\psi^{62} \otimes \left(\frac{\partial}{\partial\psi}\right)^{3} \right\|^{2/3}. \end{aligned} \tag{12.5}$$

This proves Conjecture 12.6 (B). $\square$

Remark 12.11. It seems that the families of Calabi-Yau threefolds over $\mathbb{P}^1$ studied in [31, Eqs. (2.1), (2.2)] satisfy Assumption (i), (ii), (iii) of Sect. 11. (See [31, p.157, last five lines].) By the explicit formula for the Yukawa coupling [31, Eq. (4.6)], we get $\mathcal{R} \cap (\mathbb{P}^1 \setminus \mathcal{D}) = \emptyset$ for these examples. If $\mathcal{R} \cap \mathcal{D}^* = \emptyset$, the conjectured formulas for the BCOV invariants of these families [6, p.294] follow from Theorem 11.3 (2).

13. **The BCOV invariant of FHSV threefolds**

13.1. **The threefolds of Ferrara-Harvey-Strominger-Vafa**

A compact connected complex surface S is an *Enriques surface* if it satisfies $H^1(S, \mathcal{O}_S) = 0$, $K_S \not\cong \mathcal{O}_S$, and $K_S^2 \cong \mathcal{O}_S$. An Enriques surface S is an algebraic surface with $\pi_1(S) \cong \mathbb{Z}_2$ whose universal covering $\widetilde{S}$ is a $K3$ *surface*. For an Enriques surface S, let $\iota_S \colon \widetilde{S} \to \widetilde{S}$ be the non-trivial covering transformation that generates $\pi_1(S)$. Then $(\widetilde{S}, \iota_S)$ is a 2-elementray $K3$ surface. (See [60, Sect. 8.1].)

Let $\mathbb{H} \subset \mathbb{C}$ be the complex upper-half plane. For $\tau \in \mathbb{H}$, let E_τ denote the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. For an elliptic curve $T = E_\tau$, let -1_T be the involution on T defined as $-1_T(z) = -z$ for $z \in \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Let $\mathbb{Z}_2$ be a group of order 2 with generator θ. Then $\mathbb{Z}_2$ acts on the spaces $\widetilde{S}$, T, and $\widetilde{S} \times T$ by identifying θ with ι_S, -1_T and $\iota_S \times (-1_T)$, respectively.

Definition 13.1. For an Enriques surface S and an elliptic curve T, define

$$X_{(S,T)} := \widetilde{S} \times T/\mathbb{Z}_2.$$

Since $\iota_S \times (-1_T)$ has no fixed points, $X_{(S,T)}$ is a smooth Calabi-Yau threefold. Let $p_1 \colon X_{(S,T)} \to S = \widetilde{S}/\mathbb{Z}_2$ and let $p_2 \colon X_{(S,T)} \to \mathbb{P}^1/\mathbb{Z}_2$ be the natural projections. Then p_1 is an elliptic fibration with constant fiber T, and p_2 is a $K3$ fibration

with constant fiber $\widetilde{S}$. After Ferrara-Harvey-Strominger-Vafa [19], the Calabi-Yau threefold $X_{(S,T)}$ is called the *FHSV threefold* associated with (S,T). We have

$$\chi(X_{(S,T)}) = \frac{1}{2}\,\chi(\widetilde{S}\times T) = \frac{1}{2}\,\chi(\widetilde{S})\chi(T) = 0. \tag{13.1}$$

13.2. The moduli space of FHSV threefolds

The period of an Enriques surface S is defined as the period of $(\widetilde{S},\iota_S)$ and lies in the quotient space Ω/Γ, where Ω is a symmetric bounded domain of type IV of dimension 10 and where Γ is an arithmetic subgroup of $\mathrm{Aut}(\Omega)$. The period of S is denoted by $[S]\in\Omega/\Gamma$. There exists a Γ-invariant divisor $D\subset\Omega$, called the discriminant locus, such that $(\Omega\setminus D)/\Gamma$ is a coarse moduli space of Enriques surfaces via the period map. We refer to e.g. [2, Chap. 8, Sects. 19-21] for more details about the moduli space of Enriques surfaces.

In [13], Borcherds constructed an automorphic form Φ on Ω for Γ of weight 4 with $\mathrm{div}(\Phi)=D$. The automorphic form Φ is called the *Borcherds Φ-function.* Let B_Ω be the Bergman kernel function of Ω. The Petersson norm of the Borcherds Φ-function is the Γ-invariant C^∞ function on Ω defined as

$$\|\Phi\|^2 := B_\Omega^4|\Phi|^2.$$

By the Γ-invariance of $\|\Phi\|^2$, it descends to a function on Ω/Γ, denoted again by $\|\Phi\|^2$. Then $\|\Phi([S])\|^2$ is the value of the Petersson norm of the Borcherds Φ-function at the period point of an Enriques surface S. We refer to [13], [60] for more details about the Borcherds Φ-function.

For an elliptic curve $T\cong E_\tau$, the period of T is defined as the $SL_2(\mathbb{Z})$-orbit of $\tau\in\mathbb{H}$ and is denoted by $[T]\in\mathbb{H}/SL_2(\mathbb{Z})$. The quotient space $\mathbb{H}/SL_2(\mathbb{Z})$ is a coarse moduli space of elliptic curves via the period map. Let

$$\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24}, \qquad q:=\exp(2\pi\sqrt{-1}\tau)$$

be the Jacobi Δ-function, which is a unique cusp form of weight 12. The Petersson norm of the Jacobi Δ-function is a $SL_2(\mathbb{Z})$-invariant C^∞ function on $\mathbb{H}$ defined as

$$\|\Delta(\tau)\|^2 := (\det\mathrm{Im}\,\tau)^{12}|\Delta(\tau)|^2.$$

By the $SL_2(\mathbb{Z})$-invariance of $\|\Delta\|^2$, it descends to a function on $\mathbb{H}/SL_2(\mathbb{Z})$. Then $\|\Delta([T])\|^2$ is the value of the Petersson norm of the Jacobi Δ-function at the period point of an elliptic curve T.

Theorem 13.2. *The analytic space $[(\Omega\setminus D)/\Gamma]\times[\mathbb{H}/SL_2(\mathbb{Z})]$ is a coarse moduli space of FHSV threefolds.*

Proof. Since $(\Omega\setminus D)/\Gamma$ is a coarse moduli space of Enriques surfaces [2, Chap. 8, Ths. 21.2 and 21.4] and since $\mathbb{H}/SL_2(\mathbb{Z})$ is a coarse moduli space of elliptic curves via the elliptic j-function, it suffices to prove that $X_{(S,T)}\cong X_{(S',T')}$ if and only if $S\cong S'$ and $T\cong T'$. This statement follows from [5, Sect. 3]. □

13.3. A Conjecture of Harvey-Moore

Following [25, Sect. V] and [60, Sect. 8.1], we interpret a result of the third-named author [60, Th. 8.3] in terms of the BCOV torsion of FHSV threefolds. The following formula was conjectured by Harvey-Moore [25, Eq. (4.9)].

Theorem 13.3. *There exists a constant C such that for every Enriques surface S and for every elliptic curve T,*

$$\tau_{\mathrm{BCOV}}(X_{(S,T)}) = C\,\|\Phi([S])\|^2\,\|\Delta([T])\|^2.$$

For the proof of Theorem 13.3, we need some intermediary results. Let $H^2_+(\widetilde{S},\mathbb{Z})$ be the invariant subspace of $H^2(\widetilde{S},\mathbb{Z})$ with respect to the ι_S-action. Let $H \in H^2_+(\widetilde{S},\mathbb{Z})$ be an ι_S-invariant Kähler class on $\widetilde{S}$, and let $\mathbf{v} \in H^2(T,\mathbb{Z})$ be the generator with $\int_T \mathbf{v} = 1$. Let $\pi\colon \widetilde{S}\times T \to X_{(S,T)}$ be the natural projection. We define $\kappa \in H^2(X_{(S,T)},\mathbb{Z})$ to be the Kähler class on $X_{(S,T)}$ such that $\pi^*\kappa = H+\mathbf{v}$. By [58], there exists a unique Ricci-flat Kähler form $\gamma=\gamma_\kappa$ on $X_{(S,T)}$ with Kähler class κ. By [58] again, there exist a unique Ricci-flat Kähler form γ_H on $\widetilde{S}$ and a unique Ricci-flat Kähler form γ_T on T such that

$$\pi^*\gamma_\kappa = \gamma_H + \gamma_T, \qquad [\gamma_H] = H, \qquad [\gamma_T] = \mathbf{v}.$$

Let $\langle\cdot,\cdot\rangle$ denote the cup-product pairing on $H^2(\widetilde{S},\mathbb{Z})$. Since $\int_T \mathbf{v} = 1$ and $\langle a,b\rangle = \int_{\widetilde{S}} a\wedge b$ for $a,b \in H^2(\widetilde{S},\mathbb{Z})$, we get

$$\mathrm{Vol}(X_{(S,T)},\gamma) = \frac{1}{2}\int_{\widetilde{S}\times T}\frac{(H+\mathbf{v})^3}{(2\pi)^3 3!} = \frac{1}{2^5\pi^3}\langle H,H\rangle. \tag{13.2}$$

By the Ricci-flatness of γ, Remark 4.2, and (13.1), we get

$$\mathcal{A}(X_{(S,T)},\gamma) = \mathrm{Vol}(X_{(S,T)},\gamma)^{\chi(X_{(S,T)})/12} = 1. \tag{13.3}$$

Lemma 13.4. *The following identity holds:*

$$\mathrm{Vol}_{L^2}(H^2(X_{(S,T)},\mathbb{Z}),\kappa) = \frac{\langle H,H\rangle}{2^{35}\pi^{33}}.$$

Proof. Let $H^2_+(\widetilde{S}\times T,\mathbb{Z})$ be the invariant subspace of $H^2(\widetilde{S}\times T,\mathbb{Z})$ with respect to the $\iota_S\times(-1_T)$-action. Similarly, let $H^2_+(T,\mathbb{Z})$ be the invariant subspace of $H^2(T,\mathbb{Z})$ with respect to the -1_T-action. We have

$$\pi^*H^2(X_{(S,T)},\mathbb{Z})_{\mathrm{fr}} = H^2_+(\widetilde{S}\times T,\mathbb{Z}) = H^2_+(\widetilde{S},\mathbb{Z})\oplus H^2_+(T,\mathbb{Z}) = H^2_+(\widetilde{S},\mathbb{Z})\oplus\mathbb{Z}\,\mathbf{v}. \tag{13.4}$$

By [2, Chap. 8, Lemma 15.1 (iii)], there exists an integral basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{10}\}$ of $H^2_+(\widetilde{S},\mathbb{Z})$ such that

$$\det(\langle\mathbf{e}_i,\mathbf{e}_j\rangle)_{1\le i,j\le 10} = -2^{10}. \tag{13.5}$$

By (13.4), we fix the basis $\{\bar{\mathbf{e}}_1,\dots,\bar{\mathbf{e}}_{10},\bar{\mathbf{v}}\}$ of $H^2(X_{(S,T)},\mathbb{Z})_{\mathrm{fr}}$ such that

$$\pi^*(\bar{\mathbf{e}}_i) = \mathbf{e}_i \quad (1\le i\le 10), \qquad \pi^*(\bar{\mathbf{v}}) = \mathbf{v}.$$

Recall that the cubic form $c = c_{X_{(S,T)}}$ on $H^2(X_{(S,T)},\mathbb{Z})_{\mathrm{fr}}$ was defined in Sect. 4.4. Then we get

$$c(\bar{\mathbf{e}}_i,\bar{\mathbf{v}},\kappa) = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge\mathbf{v}\wedge\pi^*\kappa = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge\mathbf{v}\wedge(H+\mathbf{v}) = \frac{1}{2(2\pi)^3}\langle\mathbf{e}_i,H\rangle,$$

$$c(\bar{\mathbf{e}}_i,\bar{\mathbf{e}}_j,\kappa) = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge\mathbf{e}_j\wedge\pi^*\kappa = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge\mathbf{e}_j\wedge(H+\mathbf{v}) = \frac{1}{2(2\pi)^3}\langle\mathbf{e}_i,\mathbf{e}_j\rangle,$$

$$c(\bar{\mathbf{e}}_i,\kappa,\kappa) = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge(\pi^*\kappa)^2 = \frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{e}_i\wedge(H+\mathbf{v})^2 = \frac{1}{(2\pi)^3}\langle\mathbf{e}_i,H\rangle,$$

$$c(\bar{\mathbf{v}},\bar{\mathbf{v}},\kappa)=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{v}\wedge\mathbf{v}\wedge\pi^*\kappa=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{v}\wedge\mathbf{v}\wedge(H+\mathbf{v})=0,$$

$$c(\bar{\mathbf{v}},\kappa,\kappa)=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{v}\wedge(\pi^*\kappa)^2=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}\mathbf{v}\wedge(H+\mathbf{v})^2=\frac{1}{2(2\pi)^3}\langle H,H\rangle,$$

$$c(\kappa,\kappa,\kappa)=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}(\pi^*\kappa)^3=\frac{1}{2(2\pi)^3}\int_{\widetilde{S}\times T}(H+\mathbf{v})^3=\frac{3}{2(2\pi)^3}\langle H,H\rangle.$$

By Lemma 4.12 and these formulae, we get

$$(2\pi)^3\langle\bar{\mathbf{e}}_i,\bar{\mathbf{e}}_j\rangle_{L^2,\kappa}=\frac{3}{2}\frac{c(\bar{\mathbf{e}}_i,\kappa,\kappa)c(\bar{\mathbf{e}}_j,\kappa,\kappa)}{c(\kappa,\kappa,\kappa)}-c(\bar{\mathbf{e}}_i,\bar{\mathbf{e}}_j,\kappa)=\frac{\langle\mathbf{e}_i,H\rangle\,\langle\mathbf{e}_j,H\rangle}{\langle H,H\rangle}-\frac{1}{2}\langle\mathbf{e}_i,\mathbf{e}_j\rangle,$$

$$(2\pi)^3\langle\bar{\mathbf{e}}_i,\bar{\mathbf{v}}\rangle_{L^2,\kappa}=\frac{3}{2}\frac{c(\bar{\mathbf{e}}_i,\kappa,\kappa)c(\bar{\mathbf{v}},\kappa,\kappa)}{c(\kappa,\kappa,\kappa)}-c(\bar{\mathbf{e}}_i,\bar{\mathbf{v}},\kappa)=\frac{1}{2}\frac{\langle\mathbf{e}_i,H\rangle\,\langle H,H\rangle}{\langle H,H\rangle}-\frac{1}{2}\langle\mathbf{e}_i,H\rangle=0,$$

$$(2\pi)^3\langle\bar{\mathbf{v}},\bar{\mathbf{v}}\rangle_{L^2,\kappa}=\frac{3}{2}\frac{c(\bar{\mathbf{v}},\kappa,\kappa)c(\bar{\mathbf{v}},\kappa,\kappa)}{c(\kappa,\kappa,\kappa)}-c(\bar{\mathbf{v}},\bar{\mathbf{v}},\kappa)=\frac{1}{4}\frac{\langle H,H\rangle\,\langle H,H\rangle}{\langle H,H\rangle}-0=\frac{1}{4}\langle H,H\rangle,$$

which yields that

$$\begin{aligned}&\mathrm{Vol}_{L^2}(H^2(X_{(S,T)},\mathbb{Z}),\kappa)\\&=\det\begin{pmatrix}\langle\bar{\mathbf{e}}_i,\bar{\mathbf{e}}_j\rangle_{L^2,\kappa}&\langle\bar{\mathbf{e}}_i,\bar{\mathbf{v}}\rangle_{L^2,\kappa}\\\langle\bar{\mathbf{e}}_i,\bar{\mathbf{v}}\rangle_{L^2,\kappa}&\langle\bar{\mathbf{v}},\bar{\mathbf{v}}\rangle_{L^2,\kappa}\end{pmatrix}\\&=(2\pi)^{-33}2^{-10}\,\frac{\langle H,H\rangle}{4}\,\det\left(\langle\mathbf{e}_i,\mathbf{e}_j\rangle-2\frac{\langle\mathbf{e}_i,H\rangle\,\langle\mathbf{e}_j,H\rangle}{\langle H,H\rangle}\right)_{1\leq i,j\leq 10}.\end{aligned}\tag{13.6}$$

Define a 10×10 symmetric matrix A by $A=(\langle\mathbf{e}_i,\mathbf{e}_j\rangle)$. Write $H=\sum_{i=1}^{10}h_i\,\mathbf{e}_i$ and define a column vector $\mathbf{h}\in\mathbb{Z}^{10}$ by $\mathbf{h}=(h_i)$. We set

$$B:=A-2\,\frac{(A\mathbf{h})\cdot({}^t\mathbf{h}A)}{{}^t\mathbf{h}A\mathbf{h}}.$$

Since A is invertible and since ${}^t\mathbf{h}\,A\,\mathbf{h}=\langle H,H\rangle>0$, we get the decomposition $\mathbb{R}^{10}=\mathbb{R}\,\mathbf{h}\oplus(A\,\mathbf{h})^{\perp}$. Since $B\,\mathbf{h}=-A\,\mathbf{h}$ and $B\,\mathbf{x}=A\,\mathbf{x}$ for $\mathbf{x}\in(A\,\mathbf{h})^{\perp}$, we get $\det B=-\det A=2^{10}$ by (13.5), which, together with (13.6), yields that

$$\mathrm{Vol}_{L^2}(H^2(X_{(S,T)},\mathbb{Z}),\kappa)=(2\pi)^{-33}2^{-10}\,\frac{\langle H,H\rangle}{4}\,\det B=\frac{\langle H,H\rangle}{2^{35}\pi^{33}}.$$

This completes the proof of Lemma 13.4. $\square$

Let $\square_H$ (resp. $\square_T$) be the $\bar{\partial}$-Laplacain of $(\widetilde{S},\gamma_H)$ (resp. (T,γ_T)) acting on $C^\infty(\widetilde{S})$ (resp. $C^\infty(T)$). We define

$$A^{\pm}(\widetilde{S}):=\{f\in C^\infty(\widetilde{S});\,\iota_S^*f=\pm f\},\qquad A^{\pm}(T):=\{f\in C^\infty(T);\,(-1_T)^*f=\pm f\}.$$

Since ι_S (resp. -1_T) preserves γ_H (resp. γ_T), $\square_H$ commutes with the ι_S-action on $C^\infty(\widetilde{S})$ and $\square_T$ commutes with the $(-1)_T$-action on $C^\infty(T)$. Hence $\square_H$ preserves $A^{\pm}(\widetilde{S})$, and $\square_T$ preserves $A^{\pm}(T)$. We set

$$\square_H^{\pm}:=\square_H|_{A^{\pm}(\widetilde{S})},\qquad\square_T^{\pm}:=\square_T|_{A^{\pm}(T)}.$$

Let $\zeta_H^{\pm}(s)$ (resp. $\zeta_T^{\pm}(s)$) be the spectral zeta function of $\square_H^{\pm}$ (resp. $\square_T^{\pm}$). Then $\zeta_H^{\pm}(s)$ and $\zeta_T^{\pm}(s)$ converges absolutely for $\mathrm{Re}\,s\gg 0$, they extend meromorphically to the complex plane $\mathbb{C}$, and they are holomorphic at $s=0$.

Lemma 13.5. *The following identity holds*

$$\log\mathcal{T}_{\mathrm{BCOV}}(X_{(S,T)},\gamma)=-24\,(\zeta_T^{+})'(0)-8\left\{(\zeta_H^{+})'(0)-(\zeta_H^{-})'(0)\right\}.$$

Proof. See [25, Sect. V], in particular [25, Eqs. (5.3), (5.9), (5.10)]. $\square$

Remark 13.6. The signs in [25, Eqs. (5.10), (5.11)] are not correct. In [25, Eqs. (5.10), (5.11)], the formula $\log\det'\square_H^{\pm} = (\zeta_H^{\pm})'(0)$ was used, while the correct formula is $\log\det'\square_H^{\pm} = -(\zeta_H^{\pm})'(0)$.

Lemma 13.7. *There exists a constant C_0 such that for every Enriques surface S and for every Kähler class H on $\widetilde{S}$, the following identity holds*

$$8\left\{(\zeta_H^+)'(0) - (\zeta_H^-)'(0)\right\} + 4\,\log\langle H,H\rangle = -\log\|\Phi([S])\|^2 + C_0.$$

Proof. The result follows from [60, Eq. (8.3)] and [62, Lemma 4.3, Eq. (4.4)]. $\square$

Lemma 13.8. *There exists a constant C_1 such that for every elliptic curve T,*

$$24\,(\zeta_T^+)'(0) = -\log\|\Delta([T])\|^2 + C_1.$$

Proof. Since $\zeta_T^+(s) = \zeta_T^-(s)$ by [46, p.166 l.8 and l.10] and since $\zeta_T^+(s)+\zeta_T^-(s)$ is the spectral zeta function of $\square_T$, the result follows from the Kronecker limit formula. See e.g. [46, Th. 4.1] or [10, Th. 13.1]. $\square$

13.4. **Proof of Theorem 13.3**

By Lemmas 13.5, 13.7, 13.8, we get

$$\log\mathcal{T}_{\rm BCOV}(X_{(S,T)},\gamma) = \log(\|\Phi([S])\|^2\,\|\Delta([T])\|^2) + 4\,\log\langle H,H\rangle - C_0 - C_1. \tag{13.7}$$

By (13.2), (13.3), (13.7) and Lemma 13.4, we get

$$\begin{aligned}
&\tau_{\rm BCOV}(X_{(S,T)},\gamma)\\
&= {\rm Vol}(X_{(S,T)},\frac{\gamma}{2\pi})^{-3}\,{\rm Vol}_{L^2}(H^2(X_{(S,T)},\mathbb{Z}),[\gamma])^{-1}\,\mathcal{A}(X_{(S,T)},\gamma)\,\mathcal{T}_{\rm BCOV}(X_{(S,T)},\gamma)\\
&= \left(\frac{\langle H,H\rangle}{2^5\pi^3}\right)^{-3}\cdot\left(\frac{\langle H,H\rangle}{2^{35}\pi^{33}}\right)^{-1}\cdot 1\cdot\frac{\|\Phi([S])\|^2\,\|\Delta([T])\|^2\langle H,H\rangle^4}{e^{C_0+C_1}}\\
&= C\,\|\Phi([S])\|^2\,\|\Delta([T])\|^2,
\end{aligned}$$

where we set $C = 2^{50}\pi^{42}\,e^{-C_0-C_1}$. This completes the proof of Theorem 13.3. $\square$

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